Stochastic Processes – 18.445

MIT, fall 2011

Practice Mid Term Exam 2

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Problem 1: . Let N_t be a birth/death process with: $\lambda_n = 4$ for $n \neq 8$, $\lambda_8 = 0$, $\mu_n = 5$ for $n \neq 9$ and $\mu_9 = 0$.

(a) Describe the communicating classes.

(b) Assuming $N_0 \leq 8$, is the process recurrent or transient? In the forrmer case find, if it exists, the equilibrium distribution. In the latter case, find if explosion occurs with positive probability.

(c) Assuming $N_0 \geq 9$, is the process recurrent or transient? In the forrmer case find, if it exists, the equilibrium distribution. In the latter case, find if explosion occurs with positive probability.

Problem 2: . A small barber shop, operated by a single barber, has room for only two costumers. Potential costumenrs arrive at a Poisson rate of 3 per hour, and the successive serving times are independent exponential random variables of mean 1/4 hour.

(a) What is the average number of costumers in the shop?

(b) What is the proportion of potential costumers that enter the shop?

(c) If the barber could work twice as fast, how much more business would he do (in average)?

Problem 3: . Let M_n, $n \geq 0$, with $M_0 = 0$, be a martingale, and let $X_n = M_n - M_{n-1}$, $n \geq 1$. Prove that $Var(M_n) = \sum_{i=0}^{n} Var(X_i)$.

Problem 4: . Let $X_1, X_2, X_3, \dots$ be independent identically distributed random variables. Let $m(t) = \mathbb{E}[e^{tX_1}] < \infty$ be the moment generating function of X_1. Show that

$$M_n = m(t)^{-n} e^{t(X_1+\cdots+X_n)}\,,\ n \geq 0\,,$$

is a martingale.

Problem 5: . Let $B_t, \geq 0$ be the standard Brownian motion. Find the probability density function of the following random variables:

(1) $|B_t|$,

(2) $|\max_{0\leq s\leq t} B_s|$

(3) $\max_{0\leq s\leq t} B_s - B_t$.

Problem 6: . Let $X_t = e^{at+bB_t}$, where B_t be the standard Brownian motion, and $a, b \geq 0$ are constants.

(1) Find the probability density function of X_t.

(2) Compute dX_t.

(3) For which values of a, b is X_t a martingale?