Stochastic Processes – 18.445

MIT, fall 2011

Practice Mid Term Exam 1

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Problem 1: .

Let $X_1, X_2, X_3, \ldots$ be a Markov chain on a finite state space $S = \{1, \ldots, N\}$ with transition matrix P. Among the following statements, say which implies which.

(a) There exists a probability distribution $\bar{\pi}$ such that $\lim_{n\to\infty} \pi P^n = \bar{\pi}$ for every probability distribution π.

(b) $\lim_{n\to\infty} P^n = \begin{bmatrix} \bar{\pi} \\ \vdots \\ \bar{\pi} \end{bmatrix}$, for some probability distribution $\bar{\pi}$.

(c) There exists a probability distribution $\bar{\pi}$ such that $\bar{\pi}P = \bar{\pi}$.

(d) 1 is an eigenvalue of P with multiplicity 1, and all other eigenvalues λ have $|\lambda| < 1$.

(e) $P_{ij} > 0$ for all $i, j \in S$.

(f) There exists $n > 0$ such that $P^n_{ij} > 0$ for all $i, j \in S$

(g) $P^n_{ij} > 0$ for all $i, j \in S$ and $n > 0$.

(h) For all $i, j \in S$ there exists $n > 0$ such that $P^n_{ij} > 0$.

(i) $X_1, X_2, X_3, \ldots$ is an irreducible Markov chain.

(j) $X_1, X_2, X_3, \ldots$ is an irreducible aperiodic Markov chain.

Solution:

$$(e) \Leftrightarrow (g) \implies (f) \Leftrightarrow (j) \implies \left\{ \begin{array}{l} (d) \implies (a) \Leftrightarrow (b) \\ (h) \Leftrightarrow (i) \end{array} \right\} \implies (c)$$

Problem 2: .

Let $X_1, X_2, X_3, \ldots$ be a Markov chain on $\mathbb{Z}$ such that $X_0 = 0$ and, conditioned on $X_n = i$, we have

$$X_{n+1} = \begin{cases} i-1 & \text{with prob. } \alpha \\ i & \text{with prob. } \beta \\ i+1 & \text{with prob. } \gamma \end{cases},$$

where $\alpha, \beta, \gamma \geq 0$ are such that $\alpha + \beta + \gamma = 1$. Let $T_k = \inf\{n \geq 0 \,|\, X_n = k\}$ be the time of first passage through k, and let $u_k(s) = \mathbb{E}[s^{T_k}]$, for $|s| < 1$.

(a) Sow that, for every $k \geq 1$, we have $u_k(s) = (u_1(s))^k$.

(b) Compute $u_1(s)$.

Solution: (a) $u_{k+1}(s) = \mathbb{E}[s^{T_{k+1}}] = \sum_{n=0}^{\infty} \mathbb{E}[s^{T_{k+1}} \,|\, T_k = n]\mathbb{P}[T_k = n] = \sum_{n=0}^{\infty} \mathbb{E}[s^{T_1} + n]\mathbb{P}[T_k = n] = \mathbb{E}[s^{T_1}] \sum_{n=0}^{\infty} s^n \mathbb{P}[T_k = n] = u_1(s)u_k(s)$. Hence, by induction, $u_k(s) = (u_1(s))^k$.

(b) $u_1(s) = \mathbb{E}[s^{T_1}] = \mathbb{E}[s^{T_1} \,|\, X_1 = -1]\mathbb{P}[X_1 = -1] + \mathbb{E}[s^{T_1} \,|\, X_1 = 0]\mathbb{P}[X_1 = 0] + \mathbb{E}[s^{T_1} \,|\, X_1 = 1]\mathbb{P}[X_1 = 1] = \mathbb{E}[s^{1+T_2}]\alpha + \mathbb{E}[s^{1+T_1}]\beta + \mathbb{E}[s^1]\gamma = \alpha s u_2(s) + \beta s u_1(s) + \gamma s = \alpha s (u_1(s))^2 + \beta s u_1(s) + \gamma s$. Hence, $\alpha s (u_1(s))^2 - (1 - \beta s) u_1(s) + \gamma s = 0$, which has solution:

$$u_1(s) = \frac{1 - \beta s \pm \sqrt{(1-\beta s)^2 - 4\alpha\beta s^2}}{2\alpha s}.$$

To conclude, we observe that the "+" solution is not good since it is > 1 for small $s > 0$ (while we know that, for $s \in (01)$, we must have $u_1(s) > 0$).

Problem 3: .

Let $X_1, X_2, X_3, \ldots$ be a Markov chain on $S = \{1, 2, \ldots, 11\}$ with transition matrix

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

(a) List all communicating classes, specifying if they are transient or recurrent classes.
(b) For each recurrent communicating class, decide whether it is aperiodic or periodic, and in the latter case, find its period.

Solution: Communicating classes:
(1) $\{1, 2, 3\}$: recurrent, aperiodic
(2) $\{4, 5\}$: transient,
(3) $\{6, 7, 8, 9, 10, 11\}$: recurrent, periodic of period 2.

Problem 4: .
Let $X_1, X_2, X_3, \ldots$ be a sequence of independent, identically distributed random variables, with values in $S = \{1, 2, 3\}$ and probability distribution $\mathbb{P}[X = 1] = 1/2$, $\mathbb{P}[X = 2] = 1/3$, $\mathbb{P}[X = 3] = 1/6$. Explain why this defines a Markov chain, and compute the transition matrix P. Compute the equilibrium distribution $\bar{\pi}$.

Solution: Since X_n are independent, the Markov property holds trivially. The transition matrix is:

$$P = \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 1/2 & 1/3 & 1/6 \\ 1/2 & 1/3 & 1/6 \end{bmatrix} .$$

The equilibrium distribution is: $\bar{\pi} = (1/2, 1/3, 1/6)$.

Problem 5: .
Costumers arrive at a certain facility according to a Poisson process of rate λ. It is known that exactly 5 costumers arrives in the first hour. Each costumer, independently from the other costumers, spends a time T in the store that is an exponential random variable of rate α, and then leave the store. Compute the probability P that the store is empty at the end of this first hour.

Solution: As we proved in class, conditioned on $X_1 = 5$, the waiting times $W_1, \ldots, W_5$ are uniformly distributed in the interval $[0, 1]$. Namely, if $S_1, \ldots, S_5$ are independent uniform random variable on $[0, 1]$, then the waiting times can be constructed by imposing that $W_1 < \cdots < W_5$ and $\{S_1, \ldots, S_5\} = \{W_1, \ldots, W_n\}$ (as unordered sets). If T_i is the time that costumer i spends in the shop, we thus have

$$P = \mathbb{P}[S_i + T_i < 1 \ \forall i] = \big(\mathbb{P}[S + T < 1]\big)^5 = \Big(\int_0^1 dt \alpha e^{-\alpha t} \mathbb{P}[S < 1 - t]\Big)^5 = \Big(\alpha \int_0^1 dt (1-t) e^{-\alpha t}\Big)^5 = \Big(1 - \frac{1}{\alpha} + \frac{e^{-\alpha}}{\alpha}\Big)^5 .$$

Problem 6: .
Let Y_n, $n = 0, 1, 2, \ldots$ be a Markov chain on $S = \{1, \ldots, N\}$ with transition matrix P, and let N_t be a Poisson process of rate λ. Consider the continuous time process $X_t = Y_{N_t}$. Argue that it is a continuous time Markov chain, and find its infinitesimal generating matrix A.

Solution: It is immediate to understand that X_t is a homogeneous Markov process. For small interval Δt, we have

$$\mathbb{P}[X_{\Delta t} = j \,|\, X_0 = i] \simeq \mathbb{P}[T_1 < \Delta t, Y_1 = j \,|\, Y_0 = i] = \mathbb{P}[T_1 < \Delta t] \mathbb{P}[Y_1 = j \,|\, Y_0 = i] = (1 - e^{-\lambda \Delta t}) p_{ij} \simeq (\lambda \Delta t + o(\Delta t^2)) p_{ij} .$$

Hence, the jump rates are: $\alpha(i,j) = \lambda p_{ij}$. We have $\alpha(i) = \sum_{j\neq i} \alpha(i,j) = \lambda \sum_{j\neq i} p_{ij} = \lambda(1-p_{ii})$. Hence, the generating matrix is:

$$A = \begin{bmatrix} \lambda(1-p_{11}) & \lambda p_{12} & \ldots & \lambda p_{1N} \\ \lambda p_{21} & \lambda(1-p_{22}) & \ldots & \lambda p_{2N} \\ & \ldots & & \\ \lambda p_{N1} & \lambda p_{N2} & \ldots & \lambda(1-p_{NN}) \end{bmatrix}.$$