Stochastic Processes – 18.445

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Mid Term Exam 2

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Exercise	Max Grade	Grade
1	6	6
2	6	6
3	6	6
4	6	6
5	6	6
Total	30	30

good job!

Problem 1: .

Let X_t, $t \geq 0$, be a continuous time Markov chain on $S = \{1, 2, 3\}$ with $X_0 = 1$ and infinitesimal generator matrix

$$A = \begin{bmatrix} -2 & 2 & 0 \\ 1 & -3 & 2 \\ 0 & 4 & -4 \end{bmatrix}.$$

(a) Compute $\frac{d}{dt}\mathbb{P}[X_t = n]\big|_{t=0}$, for $n = 1, 2, 3$.

(b) Compute $\mathbb{P}[X_t = 3 \,|\, X_0 = 1]$ in the limit for $t \to \infty$.

(c) Let $\theta^{(3)}$ be the time of first passage in state 3: $\theta^{(3)} = \inf\{t \geq 0 \,|\, X_t = 3\}$. Assuminig $X_0 = 1$, compute $\tau_1^{(3)} := \mathbb{E}[\theta^{(3)}]$, the mean passage time at 3.

Solution:

(A) By def of jump rates:

$$\frac{d}{dt}\mathbb{P}[X_t = n \,|\, X_0 = 1]\Big|_{t=0} = \alpha(1,n) = \begin{cases} -2 & n=1 \\ 2 & n=2 \\ 0 & n=3 \end{cases}$$

(B)

$$[x, y, z]\begin{bmatrix} -2 & 2 & 0 \\ 1 & -3 & 2 \\ 0 & 4 & -4 \end{bmatrix} = 0$$

$$\begin{cases} -2x + y = 0 \\ 2y - 4z = 0 \end{cases} \qquad \begin{bmatrix} \cancel{z} = \cancel{z} \\ y = 2x \end{bmatrix} \rightarrow \bar{\pi} = \left[\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right]$$

$\lim_{t\to\infty} \mathbb{P}[X_t = 3 \,|\, X_0 = 1]$

(C)

$$\begin{pmatrix} \tau_1^{(3)} \\ \tau_2^{(3)} \end{pmatrix} = -(A^{(3)})^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\begin{bmatrix} -2 & 2 \\ 1 & -3 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} -3 & -2 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/4 \\ 3/4 \end{bmatrix}$$

Answer:

(a) $\frac{d}{dt}\mathbb{P}[X_t = 1]\big|_{t=0} = \boxed{-2}$, $\frac{d}{dt}\mathbb{P}[X_t = 2]\big|_{t=0} = \boxed{2}$, $\frac{d}{dt}\mathbb{P}[X_t = 3]\big|_{t=0} = \boxed{0}$

(b) $\lim_{t\to\infty} \mathbb{P}[X_t = 3 \,|\, X_0 = 1] = \boxed{1/4}$

(c) $\tau_1^{(3)} = \boxed{5/4}$

Problem 2: .

(a) Let X_t be a birth and death process with rates

$$\lambda_n = 1 + \frac{1}{n+1}\ \forall n \geq 0\,, \quad \mu_n = 1\ \forall n \geq 1\,.$$

Decide wether the process it transient, null recurrent (i.e. recurrent with no invariant distribution), or positive recurrent (i.e. recurrent with invariant distribution).

(b) Let X_t be a birth and death process with rates

$$\lambda_n = 1 - \frac{2}{n+3}\ \forall n \geq 0\,, \quad \mu_n = 1\ \forall n \geq 1\,.$$

Decide wether the process it transient, null recurrent, or positive recurrent.

Solution:

- recurrent $\Leftrightarrow$ $\displaystyle\sum_n^{\infty} \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} < \infty$

- positive rec. $\Leftrightarrow$ $\displaystyle\sum_n \frac{\lambda_{n-1} \cdots \lambda_0}{\mu_n \cdots \mu_1} < \infty$

(A) $\displaystyle\sum_n \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} = \sum \frac{1}{(1+\frac{1}{2})(1+\frac{1}{3}) \cdots (1+\frac{1}{n})(1+\frac{1}{n+1})}$

$\displaystyle = \sum \frac{1}{\cancel{\frac{3}{2}} \cdot \cancel{\frac{4}{3}} \cdot \ldots \cdot \cancel{\frac{n+1}{n}} \cdot \frac{n+2}{\cancel{n+1}}} = \sum \frac{2}{n+2} = +\infty$ ⇝ recurrent

$\displaystyle\sum_n \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} = \sum_n (1+\tfrac{1}{1})(1+\tfrac{1}{2}) \cdots (1+\tfrac{1}{n})/1 =$

$\displaystyle = \sum_n \frac{2}{1} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{n}{n-1} \cdot \frac{n+1}{n} = \sum_n n = +\infty$ ⇝ null rec.

(B) $\displaystyle\sum_n \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} = \sum \frac{1}{(1-\frac{2}{4})(1-\frac{2}{5}) \cdots (1-\frac{2}{n+1})(1-\frac{2}{n+2})(1-\frac{2}{n+3})} =$

$\displaystyle = \sum \frac{1}{\frac{2}{\cancel{4}} \cdot \frac{3}{\cancel{5}} \cdot \frac{\cancel{4}}{\cancel{6}} \cdots \frac{\cancel{n-1}}{\cancel{n+1}} \cdot \frac{\cancel{n}}{n+2} \cdot \frac{\cancel{n+1}}{n+3}} = \sum \frac{(n+2)(n+3)}{6} = +\infty$ ⇝ rec.

$\displaystyle\sum_n \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} = \sum_n \frac{6}{(n+1)(n+2)} < \infty$ ⇝ pos rec.

Answer:

(a) X_t: **transient** \ **null recurrent** \ **positive recurrent** (circle one answer)

(b) X_t: **transient** \ **null recurrent** \ **positive recurrent**

$\varepsilon_1+\dots+\varepsilon_n$, ε_i : iid Bernoulli $\in\{\pm 1\}$ with $p=\frac{1}{2}$

Problem 3: .

In this exercise, X_n, $n = 0, 1, 2, \dots$ denotes the symmetric simple ($p = 1/2$) random walk on $\mathbb{Z}$ with $X_0 = 0$. For which value(s) of the parameter $\alpha \geq 0$ each of the following stochastic processes is a martingale?

(i) $3X_n + \alpha$,

(ii) $X_n^2 - \alpha$,

(iii) $X_n^2 - \alpha n$,

(iv) $e^{X_n - \alpha n}$,

(v) $X_{\min(\tau_\alpha, n)}$, where $\tau_\alpha = \inf\{n \geq 0 \mid X_n = \alpha\}$ (here α is a positive integer).

Solution:

(i) $\mathbb{E}[M_{n+1} \mid F_n] = \mathbb{E}[3X_{n+1} + \alpha \mid F_n] = \mathbb{E}[3X_n + \overset{0}{\cancel{3\varepsilon_{n+1}}} + \alpha \mid F_n] = M_n$ ✓

↳ marting. $\forall \alpha$

(ii) ε(iii) $\mathbb{E}[X_{n+1}^2 - \alpha_{n+1} \mid F_n] = \mathbb{E}[(X_n + \varepsilon_{n+1})^2 - \alpha_{n+1} \mid F_n]$

$$= X_n^2 + 2X_n \underbrace{\mathbb{E}[\varepsilon_{n+1} \mid F_n]}_{\mathbb{E}[\varepsilon_{n+1}] = 0} + \underbrace{\mathbb{E}[\varepsilon_{n+1}^2 \mid F_n]}_{\mathbb{E}[1] = 1} - \alpha_{n+1}$$

$$= X_n^2 + 1 - \alpha_{n+1} = M_n + \alpha_n + 1 - \alpha_{n+1}$$

⇝ need: $\alpha_{n+1} = \alpha_n + 1$

⇓

(ii) never (iii) $\alpha = 1$

(iv) $\mathbb{E}[e^{X_{n+1} - \alpha(n+1)} \mid F_n] = \mathbb{E}[e^{X_n + \varepsilon_{n+1} - \alpha(n+1)} \mid F_n] =$

$$= e^{X_n - \alpha(n+1)} \cdot \mathbb{E}[e^{\varepsilon_{n+1}}] = M_n \cdot e^{-\alpha} \cdot \frac{1}{2}(e + e^{-1})$$

$$\Leftrightarrow e^{-\alpha} \cdot \frac{1}{2}(e + e^{-1}) = 1 \Leftrightarrow e^{\alpha} = \frac{1}{2}(e + e^{-1}) \Leftrightarrow \alpha = \log \frac{e + e^{-1}}{2}$$

(v) $M_{n+1} = X_{\min(\tau_\alpha, n+1)} = \begin{cases} \overset{= M_n}{X_{\tau_\alpha}} & \text{if } \tau_\alpha \leq n \\ X_{n+1} & \text{if } \tau_\alpha > n \end{cases} = M_n \mathbb{1}_{\{\tau_\alpha \leq n\}} + X_{n+1} \mathbb{1}_{\{\tau_\alpha > n\}}$

F_n-meas.

⇓

$$\mathbb{E}[M_{n+1} \mid F_n] = M_n \mathbb{1}_{\{\tau_\alpha \leq n\}} + \underbrace{\mathbb{E}[X_{n+1} \mid F_n]}_{X_n = M_n} \mathbb{1}_{\{\tau_\alpha > n\}}$$

$= M_n$

↳ martingale $\forall \alpha$

Answer:

(i) $3X_n + \alpha$ is a martingale for $\alpha =$ $\boxed{\forall \alpha \in \mathbb{R}}$

(ii) $X_n^2 - \alpha$ is a martingale for $\alpha =$ $\boxed{\text{never}}$

(iii) $X_n^2 - \alpha n$ is a martingale for $\alpha =$ $\boxed{1}$

(iv) $e^{X_n - \alpha n}$ is a martingale for $\alpha =$ $\boxed{\log \frac{e + e^{-1}}{2}}$

(v) $X_{\min(\tau_\alpha, n)}$ is a martingale for $\alpha =$ $\boxed{\forall \alpha \in \mathbb{R}}$

Problem 4: .

A betting game goes as follows. At each time n, I roll a fair dice (with values $1, 2, \ldots, 6$), if the result of the dice is less than 6 (in this case I set $X_n = -1$), I loose \$1; if the result of the dice is 6 (in this case I set $X_n = +1$), I win \$$x$.

(a) Let M_n be the total winnings/losses at time n. Find a formula for M_n in terms of the results $X_1, \ldots, X_n \in \{\pm 1\}$ up to time n.

(b) Find the value of x for which M_n is a martingale (i.e. the betting game is "fair").

(c) Let now x be the value found in part (b). Suppose that I stop playing either when $M_n \leq -10$ (i.e. I loose all the \$10 that I had when I started playing), or when $M_n = 100$ (i.e. I win at least \$100), and let $\bar{M}$ be the total winnings/losses when I stop playing. What is $\mathbb{E}[\bar{M}]$?

Solution:

• $X_1, X_2, X_3, \ldots$ i.i.d. Bernoulli $\in \{\pm 1\}$, with
$p = \mathbb{P}[X_n = 1] = \frac{1}{6}$, $1-p = \mathbb{P}[X_n = -1] = \frac{5}{6}$

(A) $M_n = -\#\{i \leq n \mid X_i = -1\} + x\,\#\{1 \leq i \leq n \mid X_i = +1\}$

$$= \sum_{i=1}^{n} f(X_i), \quad \text{where} \quad f(a) = \begin{cases} -1 & \text{if } a = -1 \\ +x & \text{if } a = +1 \end{cases} = \frac{x+1}{2}a + \frac{x-1}{2}$$

$$= \sum_{i=1}^{n} \frac{x+1}{2}X_i + \frac{x-1}{2} = \frac{x+1}{2}\sum_{i=1}^{n} X_i + \frac{x-1}{2}n$$

(B) $$\mathbb{E}[M_{n+1} \mid F_n] = \mathbb{E}\left[\frac{x+1}{2}\sum_{i=1}^{n+1} X_i + \frac{x-1}{2}(n+1) \,\Big|\, F_n\right] =$$

$$= \frac{x+1}{2}\sum_{i=1}^{n} X_i + \frac{x+1}{2}\mathbb{E}[X_{n+1}] + \frac{x-1}{2}(n+1)$$

$$\mathbb{E}[X_{n+1}] = \frac{1}{6} - \frac{5}{6}$$

$$= M_n + \frac{x+1}{2}\left(\frac{1}{6} - \frac{5}{6}\right) + \frac{x-1}{2} = M_n \iff \frac{x+1}{2}\left(-\frac{4}{6}\right) + \frac{x-1}{2} = 0$$

$$\iff \frac{x+1}{3} = \frac{x-1}{2} \Rightarrow x = 5$$

(c) We can apply the O.S.T.: $\mathbb{E}[\bar{M}] = \mathbb{E}[M_T] = \mathbb{E}[M_0] = 0$

Answer:

(a) $M_n = \boxed{\sum_{i=1}^{n}\left(\frac{x+1}{2}X_i + \frac{x-1}{2}\right) = \frac{x+1}{2}\sum_{i=1}^{n} X_i + \frac{x-1}{2}\cdot n}$

(b) $x = \boxed{5}$

(c) $\mathbb{E}[\bar{M}] = \boxed{0}$

Problem 5: .

Consider the stochastic process $X_t = 3+2B_t$, $t \in [0,\infty)$, where B_t denotes the standard Brownian motion.

(a) Find the probability density function of the process X_t, at time t.

(b) Compute $\mathbb{P}[X_s = 1 \text{ for some } s \in [0,t]]$.

(c) Let τ_1 be the first time that $X_t \leq 1$. Find the probability density function of τ_1.

Solution:

(A) $\mathbb{P}[X_t \leq x] = \mathbb{P}[3+2B_t \leq x] = P[B_t \leq \frac{x-3}{2}]$

$$= \int_{-\infty}^{\frac{x-3}{2}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} dy$$

$$f_{X_t}(x) = \frac{d}{dx}\mathbb{P}[X_t \leq x] = \frac{1}{2}\frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-3)^2}{8t}} = \frac{1}{\sqrt{8\pi t}} e^{-\frac{(x-3)^2}{8t}}$$

$\hookrightarrow X_t = \mathcal{N}(\underbrace{3}_{\mu}, \underbrace{4t}_{\sigma^2})$

(B) $\mathbb{P}[X_s = 1 \text{ for some } s \in [0,t]] =$

$= \mathbb{P}[3+2B_s = 1 \text{ for some } 0 \leq s \leq t]$

$= \mathbb{P}[B_s = -1 \text{ for some } 0 \leq s \leq t]$ by symmetry

$= \mathbb{P}[B_s = +1 \text{ for some } 0 \leq s \leq t]$

$= \mathbb{P}[\max_{0\leq s \leq t} B_s \geq 1] \underset{\text{refl-princ.}}{=} 2\,\mathbb{P}[B_t \geq 1] = 2\int_1^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$

(C) $\mathbb{P}[\tau_1 \leq t] = \mathbb{P}[X_s \leq 1 \text{ for some } 0 \leq s \leq t] =$

$= \mathbb{P}[B_s \leq -1 \text{ for some } 0 \leq s \leq t] = \int_1^\infty \sqrt{\frac{2}{\pi t}} e^{-\frac{x^2}{2t}} dx$

$$f_{\tau_1}(t) = \frac{d}{dt}\mathbb{P}[\tau_1 \leq t] = \sqrt{\frac{2}{\pi}}\left(-\frac{1}{2}\right) t^{-\frac{3}{2}} \int_1^\infty e^{-\frac{x^2}{2t}} dt + \sqrt{\frac{2}{\pi t}} \int_1^\infty e^{-\frac{x^2}{2t}} \cdot \frac{+x^2}{2t^2} dx$$

$$= \frac{-1}{\sqrt{2\pi t^3}} \int_1^\infty e^{-\frac{x^2}{2t}} dt + \frac{1}{\sqrt{2\pi t^5}} \int_1^\infty x^2 e^{-\frac{x^2}{2t}} dx$$

Answer:

(a) $f_{X_t}(x) = \boxed{\frac{1}{\sqrt{8\pi t}} e^{-(x-3)^2/8t}}$

(b) $\mathbb{P}[X_s = 1 \text{ for some } s \in [0,t]] = \boxed{\sqrt{\frac{2}{\pi t}} \int_1^\infty e^{-\frac{x^2}{2t}} dt}$

(c) $f_{\tau_1}(t) = \boxed{\frac{d}{dt} \sqrt{\frac{2}{\pi t}} \int_1^\infty e^{-\frac{x^2}{2t}} dx}$

$$\boxed{= \frac{-1}{\sqrt{2\pi t^3}} \int_1^\infty e^{-\frac{x^2}{2t}} dx + \frac{1}{\sqrt{2\pi t^5}} \int_1^\infty x^2 e^{-\frac{x^2}{2t}} dx}$$