Stochastic Processes – 18.445

MIT, fall 2011

Mid Term Exam 2

December 8, 2011

Your Name: ____________________

Exercise	Max Grade	Grade
1	6	
2	6	
3	6	
4	6	
5	6	
Total	30	

Problem 1: .

Let X_t, $t \geq 0$, be a continuous time Markov chain on $S = \{1, 2, 3\}$ with $X_0 = 1$ and infinitesimal generator matrix

$$A = \begin{bmatrix} -2 & 2 & 0 \\ 1 & -3 & 2 \\ 0 & 4 & -4 \end{bmatrix}.$$

(a) Compute $\frac{d}{dt}\mathbb{P}[X_t = n]\big|_{t=0}$, for $n = 1, 2, 3$.

(b) Compute $\mathbb{P}[X_t = 3 \,|\, X_0 = 1]$ in the limit for $t \to \infty$.

(c) Let $\theta^{(3)}$ be the time of first passage in state 3: $\theta^{(3)} = \inf\big\{t \geq 0 \,\big|\, X_t = 3\big\}$. Assuminig $X_0 = 1$, compute $\tau_1^{(3)} := \mathbb{E}[\theta^{(3)}]$, the mean passage time at 3.

Solution:

Answer:

(a) $\frac{d}{dt}\mathbb{P}[X_t = 1]\big|_{t=0} =$ ☐ , $\frac{d}{dt}\mathbb{P}[X_t = 2]\big|_{t=0} =$ ☐ , $\frac{d}{dt}\mathbb{P}[X_t = 3]\big|_{t=0} =$ ☐

(b) $\lim_{t\to\infty} \mathbb{P}[X_t = 3 \,|\, X_0 = 1] =$ ☐

(c) $\tau_1^{(3)} =$ ☐

Problem 2: .

(a) Let X_t be a birth and death process with rates

$$\lambda_n = 1 + \frac{1}{n+1} \;\forall n \geq 0\,, \quad \mu_n = 1 \;\forall n \geq 1\,.$$

Decide wether the process it transient, null recurrent (i.e. recurrent with no invariant distribution), or positive recurrent (i.e. recurrent with invariant distribution).

(b) Let X_t be a birth and death process with rates

$$\lambda_n = 1 - \frac{2}{n+3} \;\forall n \geq 0\,, \quad \mu_n = 1 \;\forall n \geq 1\,.$$

Decide wether the process it transient, null recurrent, or positive recurrent.

Solution:

Answer:

(a) X_t: **transient** \ **null recurrent** \ **positive recurrent** (circle one answer)

(b) X_t: **transient** \ **null recurrent** \ **positive recurrent**

Problem 3: .

In this exercise, X_n, $n = 0, 1, 2, \ldots$ denotes the symmetric simple ($p = 1/2$) random walk on $\mathbb{Z}$ with $X_0 = 0$. For which value(s) of the parameter $\alpha \geq 0$ each of the following stochastic processes is a martingale?

(i) $3X_n + \alpha$,
(ii) $X_n^2 - \alpha$,
(iii) $X_n^2 - \alpha n$,
(iv) $e^{X_n - \alpha n}$,
(v) $X_{\min(\tau_\alpha, n)}$, where $\tau_\alpha = \inf\{n \geq 0 \,|\, X_n = \alpha\}$ (here α is a positive integer).

Solution:

Answer:

(i) $3X_n + \alpha$ is a martingale for $\alpha =$ ☐

(ii) $X_n^2 - \alpha$ is a martingale for $\alpha =$ ☐

(iii) $X_n^2 - \alpha n$ is a martingale for $\alpha =$ ☐

(iv) $e^{X_n - \alpha n}$ is a martingale for $\alpha =$ ☐

(v) $X_{\min(\tau_\alpha, n)}$ is a martingale for $\alpha =$ ☐

Problem 4: .

A betting game goes as follows. At each time n, I roll a fair dice (with values $1, 2, \ldots, 6$), if the result of the dice is less than 6 (in this case I set $X_n = -1$), I loose \$1; if the result of the dice is 6 (in this case I set $X_n = +1$), I win \$$x$.

(a) Let M_n be the total winnings/losses at time n. Find a formula for M_n in terms of the results $X_1, \ldots, X_n \in \{\pm 1\}$ up to time n.

(b) Find the value of x for which M_n is a martingale (i.e. the betting game is "fair").

(c) Let now x be the value found in part (b). Suppose that I stop playing either when $M_n \leq -10$ (i.e. I loose all the \$10 that I had when I started playing), or when $M_n = 100$ (i.e. I win at least \$100), and let $\bar{M}$ be the total winnings/losses when I stop playing. What is $\mathbb{E}[\bar{M}]$?

Solution:

Answer:

(a) $M_n =$ [ ]

(b) $x =$ [ ]

(c) $\mathbb{E}[\bar{M}] =$ [ ]

Problem 5: .

Consider the stochastic process $X_t = 3+2B_t$, $t \in [0,\infty)$, where B_t denotes the standard Brownian motion.

(a) Find the probability density function of the process X_t, at time t.

(b) Compute $\mathbb{P}[X_s = 1 \text{ for some } s \in [0,t]]$.

(c) Let τ_1 be the first time that $X_t \leq 1$. Find the probability density function of τ_1.

Solution:

Answer:

(a) $f_{X_t}(x) =$ [ ]

(b) $\mathbb{P}[X_s = 1 \text{ for some } s \in [0,t]] =$ [ ]

(c) $f_{\tau_1}(t) =$ [ ]

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