Stochastic Processes – 18.445

MIT, fall 2011

Mid Term Exam 1 – *Solutions*

October 27, 2011

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Exercise	Max Grade	Grade
1	5	*5*
2	5	*5*
3	5	*5*
4	5	*5*
5	5	*5*
6	5	*5*
Total	30	*30*

Problem 1: .

True / False questions. For each of the following statements just circle the letter **T**, if you think the statement is True, or the letter **F**, if you think the statement is False.

T or (**F**): If $X_1, X_2, X_3, \ldots$ is an irreducible Markov chain on a finite state space $S = \{1, \ldots, N\}$, then there is an equilibrium probability distribution $\bar{\pi}$ such that $\lim_{n\to\infty} \mathbb{P}[X_n = j \,|\, X_0 = i] = \bar{\pi}_j$ for every i.

(**T**) or **F**: If $X_1, X_2, X_3, \ldots$ is a Markov chain on a finite state space $S = \{1, \ldots, N\}$, then there is an invariant probability distribution $\bar{\pi}$ such that $\bar{\pi}P = \bar{\pi}$.

(**T**) or **F**: Suppose that P is a finite stochastic matrix such that 1 is a simple eigenvalue, and all other eigenvalues λ have $|\lambda| < 1$. Then $\lim_{n\to\infty} \mathbb{P}[X_n = j \,|\, X_0 = i]$ exists, and it is the same for all i.

(**T**) or **F**: Let X_t, $t \in [0, \infty)$, be a continuous time Markov chain with generating matrix A. If $A_{ij} > 0$ for all $i \neq j$, then $ker(A)$ has dimension 1.

T or (**F**): The Markov property for a continuous time Markov chain can be equivalently formulated by saying that, for all $s < t$, we have $\mathbb{P}[X_t = j \,|\, X_s = i] = \mathbb{P}[X_t - X_s = j - i]$.

(**T**) or **F**: Let X_t be the Poisson process of rate λ. Then, for all $s < t$, we have $\mathbb{P}[X_t = j \,|\, X_s = i] = \mathbb{P}[X_t - X_s = j - i]$.

Problem 2: .

Let $X_1, X_2, X_3, \ldots$ be a Markov chain on $S = \{1, 2, 3, 4, 5, 6\}$ with transition matrix

$$P = \begin{bmatrix} 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Compute (approximately) the following probabilities:
(a) $\mathbb{P}[X_{3000000000} = 2 \,|\, X_0 = 1]$,
(b) $\mathbb{P}[X_{3000000001} = 2 \,|\, X_0 = 1]$,
(c) $\mathbb{P}[X_{3000000002} = 2 \,|\, X_0 = 1]$.

Solution:

The chain is periodic of period $d = 3$. Indeed, its matrix has the block form: $P = \begin{bmatrix} 0 & A & 0 \\ 0 & 0 & A \\ A & 0 & 0 \end{bmatrix}$, where $A = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0 \end{bmatrix}$. Therefore $P^3 = \begin{bmatrix} A^3 & 0 & 0 \\ 0 & A^3 & 0 \\ 0 & 0 & A^3 \end{bmatrix}$ is block diagonal with diagonal blocks A^3 (which are irreducible aperiodic stochastic matrices).

The subclasses accessible in one step are as follows:

$$\{1, 2\} \to \{3, 4\} \to \{5, 6\}.$$

Therefore, $\mathbb{P}[X_n = 2 \,|\, X_0 = 1]$ is non zero only for n divisible by 3. In particular, $\mathbb{P}[X_{3000000001} = 2 \,|\, X_0 = 1] = \mathbb{P}[X_{3000000001} = 2 \,|\, X_0 = 1] = 0$, answering (b) and (c).

As for (a), we have $\mathbb{P}[X_{3000000000} = 2 \,|\, X_0 = 1] \simeq \lim_{n\to\infty} \mathbb{P}[X_{3n} = 2 \,|\, X_0 = 1] = \bar{\pi}_2$, where $\bar{\pi}$ is the (unique) equilibrium distribution for the irreducible aperiodic chain with transition matrix A^3, i.e. the unique solution of the equation $\bar{\pi} A^3 = \bar{\pi}$, or, equivalently, $\bar{\pi} A = \bar{\pi}$. This equation gives $\bar{\pi}_2 = \frac{1}{2}\bar{\pi}_1$, which has solution $\bar{\pi} = (\frac{2}{3}, \frac{1}{3})$.

In conclusion, $\mathbb{P}[X_{3000000000} = 2 \,|\, X_0 = 1] \simeq \frac{1}{3}$.

Answer:

(a) $\mathbb{P}[X_{3000000000} = 2 \,|\, X_0 = 1] =$ $\frac{1}{3}$

(b) $\mathbb{P}[X_{3000000001} = 2 \,|\, X_0 = 1] =$ 0

(c) $\mathbb{P}[X_{3000000002} = 2 \,|\, X_0 = 1] =$ 0

Problem 3: .

Let $X_1, X_2, X_3, \dots$ be a Markov chain on $S = \{1,2,3,4,5,6\}$ with transition matrix

$$P = \begin{bmatrix} 0.8 & 0.1 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

(a) Describe the communicating classes, specifying whether they are transient or recurrent classes.

(b) Compute, in the limit $n \to \infty$, the following probability: $\mathbb{P}[X_n = 6 \mid X_0 = 1]$.

Solution:

The matrix P has the form

$$P = \left[\begin{array}{c|cc} Q & \multicolumn{2}{c}{S} \\ \hline 0 & P_1 & 0 \\ & 0 & P_2 \end{array}\right],$$

where $Q = [0.8]$ if the substochastic matrix of transitions among transient states, $S = [0.1\ 0\ 0.1\ 0\ 0]$ gives the transition probabilities from transient to recurrent states, $P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is the transition matrix of the recurrent class $\{2,3\}$, which is clearly periodic of period 2, and $P_2 = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ is the transition matrix of the recurrent class $\{4,5,6\}$, which is irreducible aperiodic.

We have $\lim_{n\to\infty} \mathbb{P}[X_n = 6 \mid X_0 = 1] = \alpha_{\{4,5,6\}}(1)\bar{\pi}_6^{\{4,5,6\}}$, where $\alpha_R(1)$ is the probability that the chain, starting at $X_0 = 1$, eventually ends up in the recurrent class R, while $\bar{\pi}^R$ is the equilibrium distribution for the recurrent class R.

In our case, for obvious symmetry reasons, we have $\alpha_{\{2,3\}}(1) = \alpha_{\{4,5,6\}}(1) = \frac{1}{2}$. Alternatively, we can use the general formula: $\alpha_{\{4,5,6\}}(1) = \sum_{j=4,5,6} \left((1-Q)^{-1}S\right)_{1,j} = (1-0.8)^{-1}(0.1+0+0) = \frac{0.1}{0.2} = \frac{1}{2}$.

Moreover, $\bar{\pi}^{\{4,5,6\}}$ is solution of $\bar{\pi}^{\{4,5,6\}}P_2 = \bar{\pi}^{\{4,5,6\}}$. This equation gives $\bar{\pi}_6^{\{4,5,6\}} = \bar{\pi}_5^{\{4,5,6\}} = \frac{1}{2}\bar{\pi}_4^{\{4,5,6\}}$, which has solution $\bar{\pi}^{\{4,5,6\}} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$.

In conclusion, $\lim_{n\to\infty} \mathbb{P}[X_n = 6 \mid X_0 = 1] = \frac{1}{2}\frac{1}{4} = \frac{1}{8}$.

Answer:

(a) Communicating classes:

Transient class(es): $\{1\}$, Recurrent class(es) $\{2,3\}$, $\{4,5,6\}$

(b) $\lim_{n\to\infty} \mathbb{P}[X_n = 6 \mid X_0 = 1] =$ $\frac{1}{8}$

Problem 4: .

Let $X_1, X_2, X_3, \ldots$ be a Markov chain on $S = \{1, 2, 3\}$ with transition matrix

$$P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $T = \inf\{n \geq 0 \,|\, X_n = 3\}$ be the time of absorption at 3. Compute $\tau_1 = \mathbb{E}[T \,|\, X_0 = 1]$.

Solution:

In this chain 1 and 2 are transient states, and 3 is an absorbing state. Let $\tau_1 = \mathbb{E}[T \,|\, X_0 = 1]$, and $\tau_2 = \mathbb{E}[T \,|\, X_0 = 2]$. For obvious symmetry reasons, $\tau_1 = \tau_2$. Moreover, by first step analysis,

$\tau_1 = \mathbb{E}[T \,|\, X_0 = 1] = \mathbb{E}[T \,|\, X_1 = 1]\mathbb{P}[X_1 = 1 \,|\, X_0 = 1] + \mathbb{E}[T \,|\, X_1 = 2]\mathbb{P}[X_1 = 2 \,|\, X_0 = 1]$
$+\mathbb{E}[T \,|\, X_1 = 3]\mathbb{P}[X_1 = 3 \,|\, X_0 = 1] = (1 + \mathbb{E}[T \,|\, X_0 = 1])\frac{1}{3} + (1 + \mathbb{E}[T \,|\, X_0 = 2])\frac{1}{3} + 1 \times \frac{1}{3}$
$= (1 + \tau_1)\frac{1}{3} + (1 + \tau_2)\frac{1}{3} + \frac{1}{3} = 1 + \frac{2}{3}\tau_1\,.$

Hence, $\frac{1}{3}\tau_1 = 1$, i.e. $\tau_1 = 3$.

Answer:

$\tau_1 =$ $\frac{1}{3}$

Problem 5: .

Let X_t be a Poisson process of rate λ. Let $W_1, W_2, \dots$ be the waiting times (i.e. W_n is the time of n-th jump). Compute $\mathbb{P}[W_1, W_2, W_3 \leq 5 \,|\, X_7 = 3]$.

Solution:

As proved in class, conditioned on $X_7 = 3$, the random variables W_1, W_2, W_3 are uniformly distributed in the region $0 \leq w_1 < w_2 < w_3 \leq 7$, i.e. they have constant density $f_{W_1,W_2,W_3\,|\,X_7=3} = \frac{1}{Vol(0\leq w_1<w_2<w_3\leq 7)} = \frac{1}{7^3/3!}$. Hence,

$$\mathbb{P}[W_1, W_2, W_3 \leq 5 \,|\, X_7 = 3] = \frac{Vol(0 \leq w_1 < w_2 < w_3 \leq 5)}{Vol(0 \leq w_1 < w_2 < w_3 \leq 7)} = \frac{5^3/3!}{7^3/3!} = \frac{5^3}{7^3}\,.$$

Alternatively, if we let S_1, S_2, S_3 be i.i.d uniform random variable in $[0, 7]$, we have, conditioned on $X_7 = 3$, that $W_1 = \min(S_1, S_2, S_3)$, $W_2 = 2nd\min(S_1, S_2, S_3)$, $W_1 = \max(S_1, S_2, S_3)$. Hence,

$$\mathbb{P}[W_1, W_2, W_3 \leq 5 \,|\, X_7 = 3] = \mathbb{P}[S_1, S_2, S_3 \leq 5] = \mathbb{P}[S_1 \leq 5]^3 = \Big(\frac{5}{7}\Big)^3\,.$$

Answer:

$\mathbb{P}[W_1, W_2, W_3 \leq 5 \,|\, X_7 = 3] =$ $\boxed{\left(\frac{5}{7}\right)^3}$

Problem 6: .

A radioactive source emits particles according to a Poisson process of rate 2 particles per minute.

(a) Compute the probability p_a that the first particle appears some time after 3 minutes and before 5 minutes.

(b) Compute the probability p_b that exactly one particle is emitted in the time interval from 3 to 5 minutes.

Solution:

(a) Recall that the time intervals $T_1, T_2, \ldots$ for the jumps of the Poisson process are independent identically distributed exponential random variables of rate $\lambda = 2$. To say that the first particle appears some time after 3 minutes and before 5 minutes is the same as to say that $3 < T_1 < 5$. Hence

$$p_a = \mathbb{P}[3 < T_1 < 5] = \int_3^5 2e^{-2t} dt = -e^{-2t}|_3^5 = e^{-6} - e^{-10}.$$

(b) For be, we ask also that there are no other particles arriving in the interval $[3, 5]$, i.e. that $T_1 + T_2 > 5$. Hence

$$p_b = \mathbb{P}[3 < T_1 < 5,\, T_1 + T_2 > 5] = \int_3^5 f_{T_1}(t)\mathbb{P}[T_2 > 5 - t]dt$$
$$= \int_3^5 2e^{-2t}e^{-2(5-t)} dt = \int_3^5 2e^{-10} dt = 4e^{-10}.$$

Answer:

(a) $p_a =$ $e^{-6} - e^{-10}$ (a) $p_b =$ $4e^{-10}$

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