Stochastic Processes – 18.445

MIT, FALL 2011

Mid Term Exam 1

OCTOBER 27, 2011

Your Name: ____________________

Exercise	Max Grade	Grade
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
Total	30	

Problem 1: .

True / False questions. For each of the following statements just circle the letter **T**, if you think the statement is True, or the letter **F**, if you think the statement is False.

T or **F**: If X_1, X_2, $X_3, \ldots$ is an irreducible Markov chain on a finite state space $S = \{1, \ldots, N\}$, then there is an equilibrium probability distribution $\bar{\pi}$ such that $\lim_{n\to\infty} \mathbb{P}[X_n = j \,|\, X_0 = i] = \bar{\pi}_j$ for every i.

T or **F**: If X_1, X_2, $X_3, \ldots$ is a Markov chain on a finite state space $S = \{1, \ldots, N\}$, then there is an invariant probability distribution $\bar{\pi}$ such that $\bar{\pi}P = \bar{\pi}$.

T or **F**: Suppose that P is a finite stochastic matrix such that 1 is a simple eigenvalue, and all other eigenvalues λ have $|\lambda| < 1$. Then $\lim_{n\to\infty} \mathbb{P}[X_n = j \,|\, X_0 = i]$ exists, and it is the same for all i.

T or **F**: Let X_t, $t \in [0, \infty)$, be a continuous time Markov chain with generating matrix A. If $A_{ij} > 0$ for all $i \neq j$, then $ker(A)$ has dimension 1.

T or **F**: The Markov property for a continuous time Markov chain can be equivalently formulated by saying that, for all $s < t$, we have $\mathbb{P}[X_t = j \,|\, X_s = i] = \mathbb{P}[X_t - X_s = j - i]$.

T or **F**: Let X_t be the Poisson process of rate λ. Then, for all $s < t$, we have $\mathbb{P}[X_t = j \,|\, X_s = i] = \mathbb{P}[X_t - X_s = j - i]$.

Problem 2: .

Let $X_1, X_2, X_3, \ldots$ be a Markov chain on $S = \{1, 2, 3, 4, 5, 6\}$ with transition matrix

$$P = \begin{bmatrix} 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Compute (approximately) the following probabilities:

(a) $\mathbb{P}[X_{3000000000} = 2 \mid X_0 = 1]$,

(b) $\mathbb{P}[X_{3000000001} = 2 \mid X_0 = 1]$,

(c) $\mathbb{P}[X_{3000000002} = 2 \mid X_0 = 1]$.

Solution:

Answer:

(a) $\mathbb{P}[X_{3000000000} = 2 \mid X_0 = 1] =$ ☐

(b) $\mathbb{P}[X_{3000000001} = 2 \mid X_0 = 1] =$ ☐

(c) $\mathbb{P}[X_{3000000002} = 2 \mid X_0 = 1] =$ ☐

Problem 3: .

Let $X_1, X_2, X_3, \ldots$ be a Markov chain on $S = \{1, 2, 3, 4, 5, 6\}$ with transition matrix

$$P = \begin{bmatrix} 0.8 & 0.1 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

(a) Describe the communicating classes, specifying whether they are transient or recurrent classes.

(b) Compute, in the limit $n \to \infty$, the following probability: $\mathbb{P}[X_n = 6 \,|\, X_0 = 1]$.

Solution:

Answer:

(a) Communicating classes:

Transient class(es): [ ] , Recurrent class(es) [ ]

(b) $\lim_{n\to\infty} \mathbb{P}[X_n = 6 \,|\, X_0 = 1] =$ [ ]

Problem 4: .

Let $X_1, X_2, X_3, \ldots$ be a Markov chain on $S = \{1, 2, 3\}$ with transition matrix

$$P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $T = \inf\{n \geq 0 \,|\, X_n = 3\}$ be the time of absorption at 3. Compute $\tau_1 = \mathbb{E}[T \,|\, X_0 = 1]$.

Solution:

Answer:

$\tau_1 =$ $\boxed{}$

Problem 5: .

Let X_t be a Poisson process of rate λ. Let $W_1, W_2, \ldots$ be the waiting times (i.e. W_n is the time of n-th jump). Compute $\mathbb{P}[W_1, W_2, W_3 \leq 5 \,|\, X_7 = 3]$.

Solution:

Answer:

$\mathbb{P}[W_1, W_2, W_3 \leq 5 \,|\, X_7 = 3] = \boxed{}$

Problem 6: .

A radioactive source emits particles according to a Poisson process of rate 2 particles per minute.

(a) Compute the probability p_a that the first particle appears some time after 3 minutes and before 5 minutes.

(b) Compute the probability p_b that exactly one particle is emitted in the time interval from 3 to 5 minutes.

Solution:

Answer:

(a) $p_a =$ $\boxed{}$ (a) $p_b =$ $\boxed{}$

(page intentionally left blank)

.

(page intentionally left blank)