STOCHASTIC PROCESSES – 18.445 – MIT, FALL 2011

Day by day lecture outline and weekly homeworks

A) Lecture Outline

Suggested reading

Part 1: Random walk on $\mathbb{Z}$.

Lecture 1: thursday, september 8, 2011.

- Presentation of the course.
- The nearest neighbor random walk on $\mathbb{Z}$. Definition: $X_n = \sum_{m=1}^{n} B_m$, where $\{B_m\}_{m\geq 1}$ is a sequence of i.i.d. p-Bernoulli random variables.
- Markov property: $\mathbb{P}[X_n = a_n \,|\, X_0 = a_0, X_1 = a_1, \dots, X_{n-1} = a_{n-1}] = \mathbb{P}[X_n = a_n \,|\, X_{n-1} = a_{n-1}] = p$ if $a_n = a_{n-1}+1$, q is $a_n = a_{n-1}-1$, and 0 otherwise.
- Distribution at time n (two methods): $\mathbb{P}[X_n = k] = \binom{n}{\frac{n+k}{2}} p^{\frac{n+k}{2}} q^{\frac{n-k}{2}}$ if $k \equiv n(2)$, and 0 otherwise.
- Time of first passage at k: $T_k = \inf\{n \geq 1 \,|\, X_n = k\}$ (and $T_k = \infty$ of $X_n \neq k$ for all $n \geq 1$).

Lecture 2: tuesday, september 13.

- Distribution of the time of first passage at k: $\mathbb{P}[T_k = n] = \frac{|k|}{n}\mathbb{P}[X_n = k]$ if $k \neq 0$, and $\mathbb{P}[T_0 = n] = \frac{2}{n-1}\binom{n-1}{n/2} p^{n/2} q^{n/2}$ if n is even and 0 if n is odd.
- Definition of the moment generating function: $u_k(s) := \mathbb{E}[s^{T_k}] = \sum_{n=1}^{\infty} s^n \mathbb{P}[T_k = s]$.
- Applications: $\mathbb{P}[T_k < \infty] = \lim_{s\to 1^-} \mathbb{E}[s^{T_k}]$, $\mathbb{E}[T_k \,|\, T_k < \infty] = \frac{1}{\mathbb{P}[T_k<\infty]}\, , \lim_{s\to 1^-} \frac{d}{ds}\mathbb{E}[s^{T_k}]$.
- Reduction to $k = \pm 1$: $u_k(s) = u_1(s)^k$ for $k > 0$, $u_k(s) = u_{-1}(s)^{-k}$ for $k < 0$, and $u_0(s) = spu_{-1}(s) + squ_1(s)$.

Lecture 3: thursday, september 15.

- Computation of the moment generating functions: $u_k(s) = \left(\frac{1-\sqrt{1-4s^2pq}}{2sq}\right)^k$ for $k > 0$, $u_k(s) = \left(\frac{1-\sqrt{1-4s^2pq}}{2sp}\right)^{-k}$ for $k < 0$, and $u_0(s) = 1 - \sqrt{1-4s^2pq}$
- Definition of stochastic process. Several types of stochastic processes.
- Markov property and homogeneous Markov chains:

$$\mathbb{P}[X_n = j \,|\, X_0 = i_0, \dots X_{n-2} = i_{n-2},\, X_{n-1} = i] = \mathbb{P}[X_n = j \,|\, X_{n-1} = i_{n-1}] = P_{i,j}\, , .$$

Part 2: Markov chains.

Lecture 4: tuesday, september 20. K&T 3.1, 3.2, 3.3

- Description of a Markov chain in terms of the initial probability distribution $\pi_i = \mathbb{P}[X_0 = i]$, and the transition probability matrix $P_{ij} = \mathbb{P}[X_n = j \,|\, X_{n-1} = i]$.
- Examples: a two-state markov process, the simple random walk, and the Ehrenfest Urn model.
- The n-step matrix probability distribution: $P^{(n)}_{i,j} := \mathbb{P}[X_n = j \,|\, X_0 = i] = (P^n)_{i,j}$, and the probability distribution at time n: $\pi^{(n)}_i = \mathbb{P}[X_n = i] = (\pi P^n)_i$.

Lecture 5: thursday, september 22. K&T 3.4, 3.5

- Example 1: an irreducible Markov chain on a 2-state space $S = \{0, 1\}$, with transition matrix $P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$ (we made the computations for $\alpha = 1/2$, $\beta = 1/3$). Limiting transition matrix:

$$\lim_{n\to\infty} P^n = \begin{bmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix}.$$

Equilibrium (or limiting) behavior:

$$\lim_{n\to\infty} \pi P^n = \bar{\pi} = \left[\beta/(\alpha+\beta)\, ,\, \alpha/(\alpha+\beta)\right].$$

- Example 2: a Markov chain with absorbing states, on a 3 state space $S = \{0, 1, 2\}$, with transition matrix $P = \begin{bmatrix} 1 & 0 & 0 \\ \alpha & \beta & \gamma \\ 0 & 0 & 1 \end{bmatrix}$. Limiting transition matrix:

  $$\lim_{n \to \infty} P^n = \begin{bmatrix} 1 & 0 & 0 \\ \frac{\alpha}{1-\beta} & 0 & \frac{\gamma}{1-\beta} \\ 0 & 0 & 1 \end{bmatrix}.$$

  Absence of equilibrium. Stopping time $T = \inf\{n \geq 1 \,|\, X_n = 0 \text{ or } 2\}$, and its expected value (via first step analysis): $\mathbb{E}[T] = 1/(1-\beta)$.

Lecture 6: tuesday, september 27. K&T 4.1, 4.2

- Terminology: *absorbing* state: $\mathbb{P}[X_1 = i \,|\, X_0 = i] = 1$; *transient* state: $\mathbb{P}[\{X_n \neq i \,\forall n \geq 1\} \,|\, X_0 = i] > 0$; *recurrent* (=non transient) state: $\mathbb{P}[\{\exists n \geq 1 \text{ s.t. } X_n = i\} \,|\, X_0 = i] = 1$.
- *Irreducible* Markov chains:

  $$\forall i, j \in S \exists n \geq 1 \text{ s.t. } (P^n)_{i,j} > 0\,;$$

  irreducible aperiodic Markov chains:

  $$\exists n \geq 1 \text{ s.t. } \forall i, j \in S \text{ we have } (P^n)_{i,j} > 0\,;$$

  complete graph Markov chains: $P_{i,j} > 0 \,\forall i, j \in S$.
- Theorem of Perron Frobenious (without proof): a finite irreducible aperiodic Markov chain has a (unique) equilibrium distribution $\bar{\pi}$, i.e.:

  $$\lim_{n \to \infty} \pi P^n = \bar{\pi}\,.$$

  (Or, equivalently, $\lim_{n \to \infty} P^n = \begin{bmatrix} \bar{\pi} \\ \vdots \\ \bar{\pi} \end{bmatrix}$).

Lecture 7: thursday, september 29. K&T 4.2

- Application of the Perron Frobenious Thm: equilibrium distribution of the Markov chain with transition matrix: $P = \begin{bmatrix} 0.9 & 0.1 & 0 & 0 \\ 0.9 & 0 & 0.1 & 0 \\ 0.9 & 0 & 0 & 0.1 \\ 0.9 & 0 & 0 & 0.1 \end{bmatrix}$.
- Interpretation of the Perron Frobenious Thm in terms of eigenvalues and (left) eigenvectors of the transition matrix P.

Lecture 8: tuesday, october 4.

- Proof of the Perron Frobenious Thm.

Lecture 9: thursday, october 6. K&T 4.3

- Definition of communicating states $i \leftrightarrow j$.
- Proposition: $i \leftrightarrow j$ is an equivalence relation. Communicating classes.
- Classification of states: transient states and recurrent states.
- Decomposition of the transition matrix according to the communicating classes:

  $$P = \left[\begin{array}{c|ccc} Q & & S & \\ \hline & P_1 & & 0 \\ 0 & & \ddots & \\ & 0 & & P_s \end{array}\right],$$

  where Q gives transition probabilities between transient states, S gives transition probabilities between a transient state and a recurrent state, P_i are the transition matrices restricted to each recurrent class.

- Limiting behavior (for a finite Markov chain): $(*) = \lim_{n\to\infty} \mathbb{P}[X_n = j \,|\, X_0 = i]$. We have:
  $(*) = 0$ if j is transient,
  $(*) = 0$ if $i \in R_h, j \in R_k$ are in different recurrent classes,
  $(*) = \lim_{n\to\infty} (P_k)^n_{ij}$ if $i, j \in R_k$ are in the same recurrent class,
  $(*) = ?$ if i is transient and $j \in R_k$ is recurrent.
- We are left to consider the cases: i, j recurrent in the same recurrent class; i transient and j recurrent.

Tuesday, october 11. Columbus day

Lecture 10: thursday, october 13. K&T 4.4, 4.5

- Case 1: i, j in the same recurrent class R_k.
- Period of a state: $d = g.c.d.(J_i)$ where $J_i = \{n \geq 0 \,|\, (P^n)_{ii} > 0\}$.
- Proposition: $nd \in J_i$ for n sufficiently large.
- Proposition: the period d is the same within a communicating class.
- Theory of an irreducible Markov chain of period d: decomposition in periodic classes and limiting behavior of the chain, $\lim_{n\to\infty} P^n$.
- Case 2: i transient and $j \in R_k$ recurrent.
- Limiting behavior (in the case when R_k is aperiodic): $\lim_{n\to\infty} \mathbb{P}[X_n = j \,|\, X_0 = i] = \alpha_k(i)(\bar{\pi}^k)_j$, where $\bar{\pi}^k$ is the equilibrium distribution for P_k, and $\alpha_k(i) = \mathbb{P}[X_n \in R_k$ for large $n \,|\, X_0 = i]$.
- Computation of $\alpha_k(i)$.
- Ergodic properties of Markov chains.

Part 3: Continuous time Markov chains.

Lecture 11: tuesday, october 18. K&T 5.1, 5.2, 5.3

- Recall: the Poisson random variable X of parameter λ: $\mathbb{P}[X = n] = \frac{\lambda^n}{n!} e^{-\lambda}$. We have $\mathbb{E}[X] = Var[X] = \lambda$.
- The Poisson process X_t: it is the continuous time $t \in [0, \infty)$ stochastic process on $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ such that $X_0 = 0$ and it satisfies the following properties:
  - (i) independent increments: for $t_1 < t_2 < \cdots < t_n$, the r.v.'s $X_{t_1} - X_0, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent;
  - (ii) identically distributed increments: $X_t - X_s \sim X_{t-s}$ (in law);
  - (iii) X_t is a Poisson random variable of parameter λt.
- The Poisson process is used to counting "rare events". Equivalent description: X_t counts the number of events satisfying:
  - (i) the number of events in disjoint intervalf of time are independent;
  - (ii) the number of events in a time interval $[s, t]$ depends only on the length of the interval $t - s$;
  - (iii) rare events: if $X_t = n$, then $X_{t+\Delta t}$ is $n + 1$ approximately (for $\Delta t \to 0$) with probability $\lambda \Delta t$, n approximately with probability $1 - \lambda \Delta t$, and different form n or $n+1$ approximately with probability 0.
- Differential equation for $P_n(t) = \mathbb{P}[X_t = n]$: $\frac{dP_n(t)}{dt} = \lambda P_{n-1}(t) - \lambda P_n(t)$, $P_n(0) = \delta_{n,0}$. Solution: $P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$.
- The exponential r.v. T of rate λ: $\mathbb{P}[T > t] = e^{-\lambda t}$. Or, in terms of density function: $f_T(t) = \lambda e^{-\lambda t}$. Characterizing property: $\mathbb{P}[T > t + s \,|\, T > s] = \mathbb{P}[T > t]$.
- Description of the Poisson process X_t in terms of "exponential alarm clocks": let $T_1, T_2, T_3, \dots$ be a sequence of independent, identically distributed exponential r.v.'s of rate λ. Then $X_t = n$ for $T_1 + \cdots + T_n \leq t < T_1 + \cdots + T_{n+1}$.

Lecture 12: thursday, october 20. K&T 5.4,6.6

- Waiting times for the Poisson process: $W_n = T_1 + \cdots + T_n$. Theorem: conditioned on $X_t = n$, $W_1, \dots, W_n$ are uniformly distributed:: $f_{W_1,\dots,W_n|X_t=n}(x_1, \dots, x_n) = \frac{n!}{t^n}$ for $0 \leq x_1 < \cdots < x_n \leq t$ (and 0 otherwise).
- Continuous time Markov chain:
  - (i) Markov property: $\mathbb{P}[X_t = j \,|\, \{X_r, r \leq s\}] = \mathbb{P}[X_t = j \,|\, X_s]$ for all $s < t$;
  - (ii) time homogeneity: $\mathbb{P}[X_t = j \,|\, X_s = i] = \mathbb{P}[X_{t-s} = j \,|\, X_0 = i]$.
  - (iii) for $\Delta t \to 0$, we have $\mathbb{P}[X_{t+\Delta t} = j \,|\, X_t = i] \simeq \alpha(i, j)\Delta t$ for $j \neq i$, and $\mathbb{P}[X_{t+\Delta t} = i \,|\, X_t = i] \simeq 1 - \alpha(i)\Delta t$, where $\alpha(i, j) \geq 0$ is the jump rate from i to j, and $\alpha(i) = -\sum_{j \neq i} \alpha(i, j)$.

- Differential equation for $P^t_{ij} = \mathbb{P}[X_t = j \mid X_0 = i]$: $\frac{dP^t_{ij}}{dt} = \sum_{k \neq j} P^t_{ik}\alpha(kj) - P^t_{ij}\alpha(j)$.
- Infinitesimal generator matrix of the Markov chain: $A_{ij} = \alpha(i,j)$ for $i \neq j$, $A_{ii} = -\alpha(i) = -\sum_{j \neq i}\alpha(i,j)$.
- The differential equation for P^t has the matrix form: $\frac{dP^t}{dt} = P^t A$.
- Soution: $P^t = e^{tA}$.

Lecture 13: tuesday, october 25. K&T 6.1,6.6

- Example: two state continuous time Markov chain.
- Discrete time vs. continuous time Markov chains: analogies and differences.
- Description of a continuous time Markov chain X_t in terms of "exponential alarm clocks": let $T_1, T_2, T_3, \dots, X_0, X_1, X_2, X_3 \dots$ be sequences r.v.'s with the following properties:
  - (i) X_n is a Markov chain with transition matrix $P_{ij} = \alpha_{i,j}/\alpha(i)$ for $i \neq j$, $P_{ii} = 0$,
  - (ii) conditioned on $X_{n-1} = i$, T_n is independent of all the other variables X_m and T_m, and is an exponential random variable of rate $\alpha(i)$.

  Then $X_t = X_n$ for $T_1 + \cdots + T_n \leq t < T_1 + \cdots + T_{n+1}$.

Thursday, october 27. MID TERM EXAM 1

Lecture 14: tuesday, november 1.

- More on the equivalence between the descriptions of continuous time Markov chains in term of jump rates, and in terms of exponential "alarm clocks".
- Theorem: an irreducible, continuous time Markov chain on a finite state space has an equilibrium limiting behavior (i.e. there is NO period in continuous time).
- Mean passage time through k starting at i: $\tau^{(k)}_i$. It is given by: $\tau^{(k)} = -(A^{(k)})^{-1}\bar{1}$, where $A^{(k)}$ is the matrix obtained from A by erasing row and column k, and $\bar{1}$ is the column vector of 1's.

Lecture 15: thursday, november 3. K&T 6.1,6.2,6.3

- Continuous time Markov chains on an infinite state space. Typical questions:
  - (1) compute the transition probabilities $P^t_{ij} =$"e^{tA}_{ij}",
  - (2) recurrence or transience of an irreducible chain,
  - (3) existence of an (equilibrium) invariant distribution $\bar{\pi}$ for a recurrent chain,
  - (4) existence of the "explosion" phenomenon for a transient chain.
- Birth and Death processes, on $\mathbb{Z}_{\geq 0}$, with jump rates λ_n from $n \to n+1$, and μ_n from $n \to n-1$.
- Examples:
  - ex1. The Poisson process: $\lambda_n = \lambda$, $\mu_n = 0$.
  - ex2. Queue model: $MM1$ (with one server): $\lambda_n = \lambda$, $\mu_n = \mu$; MMk (with k servers): $\lambda_n = \lambda$, $\mu_n = n\mu$ for $n \leq k$, and $= k\mu$ for $n \geq k$; $MM\infty$ (with ∞-tly many servers): $\lambda_n = \lambda$, $\mu_n = n\mu$.
  - ex3. Population model: $\lambda_n = n\lambda$, $\mu_n = n\mu$. Variation with immigration: $\lambda_n = n\lambda + \nu$, $\mu_n = n\mu$.
  - ex4. Yule model (pure birth): $\lambda_n = n\lambda$.
  - ex5. Fast growing model (pure birth): $\lambda_n = n^2\lambda$.

Lecture 16: tuesday, november 8. K&T 6.4,6.5

- Recursive solution for the distribution at time t of a pure birth / pure death process.
- Example: for the Yule process (with $X_0 = 1$): $P_n(t) = \mathbb{P}[X_t = n] = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}$.
- Recurrence vs Transience of a B/D process: recurrence iff $\sum_1^\infty \frac{\mu_n\mu_{n-1}\dots\mu_1}{\lambda_n\lambda_{n-1}\dots\lambda_1}$ diverges.
- Invariant distribution (for a recurrent B/D process): it exists iff $\sum_0^\infty \frac{\lambda_{n-1}\dots\lambda_0}{\mu_n\dots\mu_1} < \infty$.
- Explosion phenomenon for a pure birth process: it occurs iff $\sum_1^\infty \frac{1}{\lambda_n} < \infty$.

Part 4: Martingales.

Lecture 17: thursday, november 10. K&T 2.4, 2.5

- Conditional expectations $\mathbb{E}[Y|\mathcal{F}_n]$ of a random variable Y conditioned on the knowledge of the process $X_0, X_1, X_2, \dots$ up to time n.
- Formulas in the discrete case:

$$\mathbb{E}[Y|\mathcal{F}_n](x_1, \dots, x_n) = \sum_y y \frac{\mathbb{P}[X_1 = x_1, \dots, X_n = x_n, Y = y]}{\mathbb{P}[X_1 = x_1, \dots, X_n = x_n]} ,$$

and in the continuous case:

$$\mathbb{E}[Y|\mathcal{F}_n](x_1,\dots,x_n) = \int_{\mathbb{R}} y \frac{f_{X_1,\dots,X_n,Y}[x_1,\dots,x_n,y]}{f_{X_1,\dots,X_n}(x_1,\dots,x_n)} dy\,.$$

- Properties:
  - (i) if Y is $\mathcal{F}_n$-measurable, then $\mathbb{E}[Y|\mathcal{F}_n] = Y$,
  - (ii) if Y is independent of $X_1,\dots,X_n$, then $\mathbb{E}[Y|\mathcal{F}_n] = \mathbb{E}[Y]$,
  - (iii) if Y is $\mathcal{F}_n$-measurable and Z is any r.v., then $\mathbb{E}[YZ|\mathcal{F}_n] = Y\mathbb{E}[Z|\mathcal{F}_n]$,
  - (iv) if $m \le n$, then $\mathbb{E}[\mathbb{E}[Y|\mathcal{F}_n]|\mathcal{F}_m] = \mathbb{E}[Y|\mathcal{F}_m]$,
  - (v) linearity: $\mathbb{E}[aY + bZ|\mathcal{F}_n] = a\mathbb{E}[Y|\mathcal{F}_n] + b\mathbb{E}[Z|\mathcal{F}_n]$, for every constants a, b.

Lecture 18: tuesday, november 15. K&T 2.5

- Definition of *Martingale* (or "model of fair game"): a stochastic process $M_0, M_1, M_2, \dots$ such that $\mathbb{E}[M_{n+1}\,|\,\mathcal{F}_n] = M_n$.
- Example: martingale betting strategy.

Lecture 19: thursday, november 17.

- Definition of *Stopping Time*: T, a random variable, such that the event $\{T > n\}$ is $\mathcal{F}_n$-measurable for every n.
- Optional Sampling Theorem: if $\mathbb{P}[T < \infty] = 1$, $\mathbb{E}[|M_T|\mathbb{1}_{T>N}] \overset{N\to\infty}{\longrightarrow} 0$, and $\mathbb{E}[|M_N|\mathbb{1}_{T>N}] \overset{N\to\infty}{\longrightarrow} 0$, then $\mathbb{E}[M_T\,|\,\mathcal{F}_0] = M_0$.
- Example: gambler's ruin: in the symmetric random walk with absorbing barriers at 0 and N and $X_0 = a$, we have $\mathbb{P}[X_\infty = N] = \frac{M}{N}$.

Part 5: Brownian motion and Ito calculus.

Lecture 20: tuesday, november 22. K&T 8.1

- Martingale Convergence Theorem:
  - (a) If $M_0, M_1, M_2, \dots$ is a martingale such that $\mathbb{E}[|M_n|] < C$ for every n, then there exists a random variable M_∞ such that $\lim_{n\to\infty} M_n = M_\infty$ with probability 1;
  - (b) If moreover $\mathbb{E}[|M_n|^2] < C$ for every n, then $\mathbb{E}[M_\infty] = \mathbb{E}[M_0]$.
- Example: the random harmonic series $\sum_{n=1}^{\infty} \frac{\pm 1}{n}$ converges with probability 1.
- The *Brownian Motion*: $\{X_t\,,\ t \in [0,\infty)\}$, such that:
  - (i) $X_0 = 0$ and X_t is a continuous function of t;
  - (ii) independent increments: for $0 < t_1 < \dots < t_n$, the random variables $X_{t_1},\ X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent;
  - (iii) time homogeneity: for $s < t$ we have $X_t - X_s \sim X_{t-s}$ (in distribution);
  - (iv) for small Δt, the r.v. $X_{\Delta t}$ has $\mathbb{E}[X_{\Delta t}] \simeq 0$, $Var(X_{\Delta t}) \simeq \sigma^2 \Delta t$.
- Distribution at time t: X_t is a Gaussian r.v. of mean 0 and variance $\sigma^2 t$.

Thursday, november 24. Thanksgiving holiday

Lecture 21: tuesday, november 29. K&T ???

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Lecture 22: thursday, december 1. K&T ???

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Lecture 23: tuesday, december 6. K&T ???

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Thursday, december 8. MID TERM EXAM 2

Lecture 24: thursday, december 10. K&T ???

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B) Weekly Homeworks

Problem Set 1 (due thursday, september 15)

Exercise 1 (K&T 1.1 p.61)**.** I roll a six-sided die and observe the number N on the upper face. I then toss a fair coin N times and I observe head appear X times.

(i) What is the probability that $N = 3$ and $X = 2$?
(ii) What is the probability that $X = 5$?
(iii) Compute $E[X]$, the expected number of heads to appear.

Exercise 2 (K&T 1.4 p.62)**.** A six-sided die is rolled and the number N on the upper face is recorded. From a jar containing 10 tags numbered $1, 2, \dots, 10$ we then select N tags at random without replacement. Let X be the smallest number on the drawn tags. Determine $P[X = 2]$.

Exercise 3 (K& T 1.2 p.62)**.** A card is picked at random from N cards labeled $1, 2, \dots, N$, and the number that appears is X. A second card is picked at random from the cards numbered $1, 2, \dots, X$, and its number is Y. Determine the conditional distribution of X given $Y = y$, for $y = 1, 2, \dots$.

Exercise 4 (K& T 1.5 p.63)**.** A nickel is tossed 20 times in succession. Every time that the nickel comes up heads, a dime is tossed. Let X count the number of heads appearing on the tosses of the dime. Determine $P[X = 0]$.

Exercise 5 (K& T 1.3 p.100)**.** Consider a sequence of items from a production process, with each item being graded as good or defective. Suppose that a good item is followed by another good item with probability α and is followed by a defective item with probability $1 - \alpha$. Similarly a defective item is followed by another defective item with probability β and is followed by a good item with probability $1 - \beta$. If the first item is good, what is the probability that the first defective item to appear is the fifth one?

Problem Set 2 (due thursday, september 22)

Exercise 6. In class we showed that, in the nearest neighbor random walk on $\mathbb{Z}$, $\{X_n\}_{n\geq 1}$, the time T_0 of first return to 0 has the following probability distribution:

$$\mathbb{P}[T_0 = n] = \frac{2}{n-1}\binom{n-1}{n/2} p^{n/2} q^{n/2} .$$

Prove, by a direct computation, that

$$\mathbb{E}[s^{T_0}] = 1 - \sqrt{1 - 4s^2pq} .$$

(Note: in class we proved this identity with an indirect argument).

(*Hint:* You may find useful the following identity: $\frac{1}{2n-1}\binom{2n-1}{n} = 2(-4)^{n-1}\binom{1/2}{n}$, that you are supposed to prove. Recall that, for any $\alpha \in \mathbb{R}$, we define $\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}$, and we have the binomial expansion $(1+x)^\alpha = \sum_{n=0}^{\infty}\binom{\alpha}{n}x^n$, for $|x| < 1$).

Exercise 7. The Smiths receive the paper every morning and place it on a pile after reading it. Each afternoon, with probability 1/3, someone takes all the papers in the pile and puts them in the recycling bin. Also, if there are 5 papers in the pile, Mr. Smith (with probability 1) takes the papers to the bin. Consider the number of papers X_n in the pile in the evening of day n. Is it reasonable to model this by a Markov chain? If so, what are the state space and the transition matrix?

Exercise 8. Consider a Markov chain with state space $\{0, 1\}$ and transition matrix

$$P = \begin{bmatrix} 1/3 & 2/3 \\ 3/4 & 1/4 \end{bmatrix} .$$

Assuming that the chain starts in state 0 at time $n = 0$, what is the probability that it is in state 1 at time $n = 2$?

Exercise 9 (K&T 1.5 p.99)**.** A Markov chain $X_0, X_1, X_2, \dots$ has the transition probability matrix (for the states {0,1,2}):

$$P = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{bmatrix} ,$$

and initial distribution: $p_0 = 0.5$, $p_1 = 0.5$, $p_2 = 0$. Determine the probabilities:

(1) $\mathbb{P}[X_0 = 1, X_1 = 1, X_2 = 0]$,
(2) $\mathbb{P}[X_0 = 1, X_1 = 1, X_3 = 0]$.

Exercise 10 (K&T 1.4 p.100)**.** The random variables $\xi_1, \xi_2, \dots$ are independent identically distributed, with common probability distribution

$$\mathbb{P}[\xi = 0] = 0.1\,,\ \mathbb{P}[\xi = 1] = 0.3\,,\ \mathbb{P}[\xi = 2] = 0.2\,,\ \mathbb{P}[\xi = 3] = 0.4\,.$$

Set $X_0 = 0$ and $X_n = \max(\xi_1, \dots, \xi_n)$ be the largest ξ observed to date. Determine the transition probability matrix for the Markov chain $\{X_n\}$.

PROBLEM SET 3 (DUE THURSDAY, SEPTEMBER 29)

Exercise 11 (K&T 2.5 p.105)**.** A Markov chain $X_n \in \{0, 1, 2\}$, starting from $X_0 = 0$, has the transition probability matrix

$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $T = \inf\{n \geq 0 \,|\, X_n = 2\}$ be the first time that the process reaches state 2, where it is absorbed. If in some experiment we observed such a process and noted that absorption has not taken place yet, we might be interested in the conditional probability that the process is in state 0 (or 1), given that absorption has not taken place. Determine $\mathbb{P}[X_3 = 0 \,|\, T > 3]$

(*Hint:* The event $\{T > 3\}$ is the same as $\{X_3 \neq 2\} = \{X_3 = 0\} \cup \{X_3 = 1\}$).

Exercise 12 (K&T 3.8 p.115)**.** Two urns A and B contain a total of N balls. Assume that at time t there were exactly k balls in A. At time $t + 1$ an urn is selected at random in proportion to its content (i.e. A is selected with probability $\frac{k}{n}$ and B with probability $\frac{N-k}{N}$). Then a ball is selected from A with probability p and from B with probability $q = 1 - p$ and placed in the previously chosen urn. Determine the transition probability matrix for the Markov chain X_t = number of balls in urn A at time t.

Exercise 13 (K&T 4.4 p.131)**.** Consider the Markov chain $X_n \in \{0, 1, 2, 3\}$ starting with state $X_0 = 1$ and with the following transition probability matrix:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.6 & 0.1 \\ 0.2 & 0.2 & 0.3 & 0.3 \end{bmatrix}.$$

Determine the probability that the process never visits state 2.

Exercise 14 (K&T 4.15 p.134)**.** A simplified model for the spread of a rumor goes this way: there are $N = 5$ people in a group of friends, of which some have heard the rumor and others have not. During any single period of time two people are selected at random from the group and assumed to interact. The selection is such that an encounter between any pair of friend is just as likely as any other pair. If one of these persons has heard the rumor and the other has not, then with probability $\alpha = 0.1$ the rumor is transmitted. Let X_n be the number of friends who have heard the rumor at time n.

Assuming that the process begins at time 0 with single person knowing the rumor, what is the mean time that it takes for everyone to hear it?

Exercise 15 (K&T 4.17 p.134)**.** The damage $X_n \in \{0, 1, 2\}$ of a system subjected to wear is a Markov chain with transition probability matrix

$$P = \begin{bmatrix} 0.7 & 0.3 & 0 \\ 0 & 0.6 & 0.4 \\ 0 & 0 & 1 \end{bmatrix}.$$

The system starts in state 0 and it fails when it first reaches state 2. Let $T = \min\{n \geq 0 \,|\, X_n = 2\}$ be the time of failure. Evaluate the moment generating function $u(s) = \mathbb{E}[s^T]$ for $0 < s < 1$.

Problem Set 4 (due thursday, october 6)

Exercise 16. Consider a Markov chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$P = \begin{bmatrix} 0.2 & 0.4 & 0.4 \\ 0.1 & 0.5 & 0.4 \\ 0.6 & 0.3 & 0.1 \end{bmatrix}.$$

Compute the probability that, in the long run, the chain is in state 1 (does the answer depend on the initial state X_0?). Solve this problem in two different ways:

(a) by computing the matrix P^n and letting $n \to \infty$;
(b) by finding the (unique) invariant probability distribution as a left eigenvector of P.

Exercise 17. Consider a Markov chain with state space $S = \{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{bmatrix} 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(1) Is this Markov chain irreducible and/or aperiodic?
(2) Compute (approximately) $\mathbb{P}[X_{1000} = 1 \mid X_0 = 2]$, $\mathbb{P}[X_{1000} = 2 \mid X_0 = 2]$ and $\mathbb{P}[X_{1000} = 4 \mid X_0 = 2]$.

Exercise 18 (K&T 1.4 p.209)**.** A Markov chain $X_0, X_1, X_2, \dots$ in the state space $S = \{0, 1, 2\}$ has transition probability matrix

$$P = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}.$$

Every period that the process spends in state 0 incurs a cost of \$2, every period that the process spends in state 1 incurs a cost of \$5, every period that the process spends in state 2 incurs a cost of \$3. In the long run, what is the cost per period associated with this Markov chain?

Exercise 19 (K&T 1.7 p.210)**.** A Markov chain $X_0, X_1, X_2, \dots$ in the state space $S = \{0, 1, 2, 3\}$ has transition probability matrix

$$P = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0 & 0.3 & 0.3 & 0.4 \\ 0 & 0 & 0.6 & 0.4 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Determine the corresponding equilibrium distribution.

Exercise 20 (K&T 1.3 p.211)**.** A Markov chain $X_0, X_1, X_2, \dots$ in the state space $S = \{0, 1, 2, 3, 4, 5\}$ has transition probability matrix

$$P = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

where $\alpha_i > 0$ and $\sum_i \alpha_i = 1$. Determine, in the long run, the probability of being in state 0 (Does it depend on the initial state X_0?).

Problem Set 5 (due thursday, october 20)

Exercise 21. Let P be an arbitrary stochastic matrix, and let $v = (v_1, \dots, v_N) \in \mathbb{C}^N$ be a left eigenvector with eigenvalue 1 (explain why such v always exists).

(i) Prove that $v^+ = (|v_1|, \dots, |v_N|)$ is again a left eigenvector with eigenvalue 1 for P.
(ii) Deduce that any finite Markov chain admits an invariant probability distribution (which is not necessarily an equilibrium (=limiting) distribution).

(iii) Provide an example of a finite Markov chain with an invariant distribution $\bar{\pi}$ (such that $\bar{\pi}P = \bar{\pi}$), which is not an equilibrium distribution (namely $\lim_{n\to\infty} \pi P^n$ is not $\bar{\pi}$ for some initial distribution π).

Exercise 22. Let P be a finite stochastic matrix with the property that, for some $n \geq 1$ and some j_0, we have

$$(P^n)_{ij_0} > 0 \text{ for all } i\,, \tag{1}$$

(i.e. the j_0 column of P^n has all strictly positive entries). Prove that (almost) all conclusions of the Perron Frobeious Theorem hold:

(i) 1 is a simple eigenvalue,

(ii) its left eigenvector $\bar{\pi}$ can be chosen with non-negative entries (or, better, a probability distribution),

(iii) $\bar{\pi}$ is an equilibrium distribution: $\lim_{n\to\infty} \pi P^n = \bar{\pi}$ for all initial distributions π.

(iv) Find an example when (1) holds, but the equilibrium distribution $\bar{\pi}$ does not have all the entries strictly positive.

(v) Prove that, instead, if, for some $n \geq 1$, we have $P^n_{ij} > 0$ for all i, j, then all the entries of the equilibrium distribution $\bar{\pi}$ are strictly positive.

Exercise 23. Consider the Markov chain $X_0, X_1, X_2, \ldots$ in the state space $S = \{1,2,3,4,5,6,7,8,9,10\}$ with transition probability matrix

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2/3 & 0 \end{bmatrix}.$$

(a) Draw a graph of this Markov chain (it might be convenient to rearrange the states appropriately).

(b) Find the communicating classes, explaining which are the transient classes and which are the recursive classes (and which are absorbing states).

(c) Draw the corresponding oriented tree among the communicating classes.

(d) For each recurrent class R, find whether it is aperiodic or periodic, and in the latter case compute the period d.

(e) After rearranging the states, write the new transition matrix P in the "canonical" form

$$P = \left[\begin{array}{c|ccc} Q & & S & \\ \cline{1-4} & P_1 & & 0 \\ 0 & & \ddots & \\ & 0 & & P_s \end{array}\right],$$

where Q gives transition probabilities between transient states, S gives transition probabilities between a transient state and a recurrent state, P_i are the transition matrices restricted to each recurrent class.

(f) Compute the limiting probability $\lim_{n\to\infty} \mathbb{P}[X_n = 4 \,|\, X_0 = 2]$.

Exercise 24 (K&T 4.3 p.256)**.** Consider a random wlak Markov chain on states $\{0, 1, \ldots, N\}$ with transition probability matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ q_1 & 0 & p_1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & q_2 & 0 & p_2 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & q_3 & 0 & p_3 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots q_{N-1} & 0 & p_{N-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots 0 & 1 & 0 \end{bmatrix},$$

where $p_i > 0$, $q_i > 0$, $p_i + q_i = 1$ for all $i = 1, \ldots, N-1$. (The transition probabilities in the boundary states 0 and N "reflect" the process back into the states $1, \ldots, N-1$).

(a) Determine the invariant distribution.
(b) State whether the chain is aperiodic or periodic, and in the latter case determine its period.

Exercise 25. This week “only” four!

Problem Set 6 (due thursday, october 27)

Exercise 26 (K&T 1.3-4 p.276)**.** Recall that the *generating function* of a probability mass function $\mathbb{P}[X = k] = p_k$, $k = 0, 1, 2, \dots$, is given by

$$g_X(s) = E[s^X] = \sum_{k=0}^{\infty} p_k s^k \qquad |s| < 1\,.$$

(i) Show that the generating function of a Poisson random variable X with mean μ is $g_X(s) = e^{-\mu(1-s)}$.
(ii) Show that, if X and Y are independent, Poisson random variables with parameters α and β respectively, then the generating function of their sum is $g_{X+Y}(s) = e^{-(\alpha+\beta)(1-s)}$.

Exercise 27 (K&T 1.7 p.277)**.** Shocks occur to a system according to a Poisson process of rate λ. Suppose that the systems survives each shock with probability α, independently of other shocks, so that its probability of surviving k shocks is α^k. What is the probability that the system is surviving at time t?

Exercise 28 (K&T 2.4,p.288)**.** Suppose that N points are uniformly distributed over the interval $[0, N)$. Determine the probability distribution for the number of points in the interval $[0, 1)$ as $n \to \infty$.

Exercise 29 (K&T 3.1-4 p.295)**.** (i) Let X_t be a Poisson process of rate λ. Show that

$$W_1 > w_1\,,\; W_2 > w_2 \Longleftrightarrow X_{w_1} = 0\,,\; X_{w_2} - X_{w_1} = 0 \text{ or } 1\,.$$

(ii) Determine the joint upper tail probability:

$$\mathbb{P}[W_1 > w_1, W_2 > w_2] = e^{-\lambda w_1}\big(1 + \lambda(w_2 - w_1)\big)e^{-\lambda(w_2 - w_q)}\,.$$

(iii) Differentiate twice to determine the joint density function

$$f(w_1, w_2) = \lambda^2 e^{-\lambda w_2}\,.$$

(iv) Consider the variables $T_1 = W_1$, $T_2 = W_2 - W_1$. Determine the joint probability distribution of T_1 and T_2.

Exercise 30 (K&T 6.4 p.407)**.** A system consists of two units, both of which may operate simultaneously, and a single repair facility. The probability that an operating system will fail in a short time interval of lenght Δt is $\mu\Delta t + o(\Delta t)$. Repair times are exponentially distributed, but the parameter depends whether the failure was *regular* or *severe*. The fraction of regular failures is p, and the corresponding exponential parameter is α. The fraction of severe failures is $q = 1-p$, and the corresponding exponential parameter is $\beta < \alpha$.

Model the system as a continuous time Markov chain, by taking as states the pairs (x, y), where $x = 0, 1, 2$ is the number of units operating, and $y = 0, 1, 2$ is the number of units undergoing repair of a severe failure. The possible states are $(2, 0)$, $(1, 0)$, $(1, 1)$, $(0, 0)$, $(0, 1)$, and $(0, 2)$. Specify the infinitesimal matrix A. Assume that the units enter the repair shop on a first come first serve basis.

Problem Set 7 (due thursday, november 3)

Exercise 31. Let N_t be a continuous time Markov chain with values in $\mathbb{N} = \{0, 1, 2, \dots\}$, with $N_0 = 0$ and jump rates: $\alpha(n, n+1) = \alpha_0$ if n is even, $\alpha(n, n+1) = \alpha_1$ if n is odd, and $\alpha(n, n') = 0$ for $n' \neq n, n+1$. Let $P_0(t) = \mathbb{P}[N_t \text{ is even}]$ and $P_1(t) = \mathbb{P}[N_t \text{ is odd}]$.

(a) derive a system of linear differential equations (of first order) for $P_0(t)$ and $P_1(t)$;
(b) solve this system to compute $P_0(t)$ and $P_1(t) = 1 - P_0(t)$.

Exercise 32. Let X_t be a continuous time Markov chain on a finite state space $S = \{1, \dots, N\}$, with infinitesimal generator matrix A.

(a) show that a row vector $v = (v_1, \dots, V_N) \in \mathbb{R}^N$ satisfies $ve^{tA} = v$ for all $t \geq 0$ if and only if $vA = 0$.
(b) Deduce that a probability distribution $\bar{\pi}$ is *invariant* in time (i.e. if X_0 has distribution $\bar{\pi}$, so does X_t for all t) if and only if $\bar{\pi}A = 0$.
(c) show that, if $\bar{\pi}$ is an *equilibrium* distribution (in the sense that $\mathbb{P}[X_t = j] \to \bar{\pi}_j$ for $t \to \infty$), then $\bar{\pi}$ must be an invariant distribution, i.e. $\bar{\pi}A = 0$.

In the following exercises 33 and 34 we want to understand the equivalence of the two constructions of continuous time Markov chains explained in class.

*R*ecall: that a continuous time Markov chain on $S = \{1, \dots, N\}$ with infinitesimal jump rates $\alpha(i,j)$, $i, j \in S$, is defined by the following three **properties:**

(i) Markov property: $\mathbb{P}[X_t = j \,|\, \{X_r = i_r\, r \leq s\}] = \mathbb{P}[X_t = j \,|\, X_s = i_s]$ for all $s < t$;
(ii) time homogeneity: $\mathbb{P}[X_t = j \,|\, X_s = i] = \mathbb{P}[X_{t-s} = j \,|\, X_0 = i]$ for all $s < t$;
(iii) jump rates: $\mathbb{P}[X_{\Delta t} = j \,|\, X_0 = i] = \alpha(i,j)\Delta t + o(\Delta t)$ for $j \neq i$, and $\mathbb{P}[X_{\Delta t} = i \,|\, X_0 = i] = 1 - \alpha(i)\Delta t + o(\Delta t)$, where $\alpha(i) = \sum_{j \neq i} \alpha(i,j)$.

Given the continuous time Markov chain X_t, we construct two sequences of random variables:

$$\begin{array}{ll} T_1 = \inf\{t \geq 0 \,|\, X_t \neq X_0\}\ , & X_1 = X_{T_1}\ , \\ T_2 = \inf\{t \geq 0 \,|\, X_{t+T_1} \neq X_1\}\ , & X_2 = X_{T_1+T_2}\ , \\ \dots & \\ T_n = \inf\{t \geq 0 \,|\, X_{t+T_1+\dots+T_{n-1}} \neq X_{n-1}\}\ , & X_n = X_{T_1+\dots+T_n}\ . \end{array}$$

We want to understand the following results (stated in class):

Theorem 1: if X_t is a continuous time Markov chain, then: $X_0, X_1, X_2, \dots$ form a (discrete time) Markov chain with transition matrix $P = \left(p_{ij}\right)_{i,j=1}^N$ given by $p_{ii} = 0$ for all i, and $p_{ij} = \alpha(i,j)/\alpha(i)$ for all $i \neq j$; $T_1, T_2, \dots$ are independent random variables, and, conditioned on $X_{n-1} = i$, T_n is an exponential random variable of rate $\alpha(i)$.

Exercise 33. Suppose that X_t is a continuous time Markov chain with jump rates $\alpha(i,j)$ (i.e. it satisfies the properties (i), (ii), (iii) above).

(a) use the Markov property and time homogeneity of X_t to deduce that $\mathbb{P}[X_\tau = i\, \forall \tau \leq s + t \,|\, X_\sigma = i\, \forall \sigma \leq s] = \mathbb{P}[X_\tau = i\, \forall \tau \leq t \,|\, X_0 = i]$;
(b) deduce that $\mathbb{P}[T_1 > t + s \,|\, T_1 > s, X_0 = i] = \mathbb{P}[T_1 > t \,|\, X_0 = i]$;
(c) observe that the only functions satisfying the identity $f(x+y) = f(x)f(y)$ for all x, y are: $f(x) = Ce^{kx}$, for some constants C, k;
(d) deduce that $\mathbb{P}[T_1 > t \,|\, X_0 = i] = e^{-\lambda t}$ for some positive constant λ, i.e., conditioned on $X_0 = i$, T_1 must be an exponential random variable of a certain rate λ;
(e) from the fact that $\mathbb{P}[X_{\Delta t} = i \,|\, X_0 = i] \sim 1 - \alpha(i)\Delta t$, deduce that it must be $\lambda = \alpha(i)$, i.e. T_1 is exponential random variable of rate $\alpha(i)$;

Exercise 34. Again, suppose that X_t is a continuous time Markov chain (as defined by properties (i), (ii), (iii) above), and let $T_1, T_2, \dots, X_0, X_1, X_2, \dots$ be defined in terms of X_t as above. We want to show that Theorem 1 holds. For this, prove that

$$\begin{aligned} & \mathbb{P}[T_1 \in (t_1, t_1 + \Delta t_1), T_2 \in [t_2, t_2 + \Delta t_2], X_0 = i, X_1 = j, X_2 = k] \\ & \simeq \mathbb{P}[T_1 \in (t_1, t_1 + \Delta t_1) \,|\, X_0 = i] \mathbb{P}[T_2 \in (t_2, t_2 + \Delta t_2) \,|\, X_1 = i] \mathbb{P}[X_0 = i] p_{ij} p_{jk} \\ & \simeq \alpha(i) e^{\alpha(i) t_1} \alpha(j) e^{\alpha(j) t_2} \Delta t_1 \Delta t_2 \mathbb{P}[X_0 = i] \alpha(i,j) \alpha(j,k)\,. \end{aligned}$$

(Note: for this you should express everything in terms of the continuous time chain X_t, and use its three defining properties (i), (ii), (iii) (by considering appropriate disjoint time intervals) to compute the desired probability)

Next, in exercise 35 we want to prove a "converse statement" to Theorem 1:

Theorem 2: Let $X_0, X_1, X_2, \dots$ be a (discrete time) Markov chain with transition matrix $P = \left(p_{ij}\right)_{i,j=1}^N$, and let $T_1, T_2, \dots$ be a sequence of independent identically distributed random variables of rate λ. Assume that the sequence of random variables $\{X_n\}$ and the sequence $\{T_n\}$ are independent from each other. Construct the following continuous time stochastic process:

$$X_t = X_n \quad \text{for } T_1 + \dots + T_n \leq t < T_1 + \dots + T_{n+1}\,.$$

Then: X_t is a continuous time Markov chain of jump rates $\alpha(i,j) = \lambda p_{ij}$.

Exercise 35. We want to prove that Theorem 2 holds. For this, show that the process X_t constructed above satisfies the following property:

$$\begin{aligned} & \mathbb{P}[X_{t+\Delta t} = j \,|\, \{X_s = i_s\, \forall s \leq t\}] = \mathbb{P}[X_{t+\Delta t} = j \,|\, X_t = i = i_t] \quad \text{(i.e. Markov property (i) holds)} \\ & = \mathbb{P}[X_{\Delta t} = j \,|\, X_0 = i] \quad \text{(i.e. time homogeneity property (ii) holds)} \\ & \simeq \left\{ \begin{array}{l} \lambda p_{ij} \Delta t \text{ for } j \neq i \\ 1 - \lambda(1 - p_{ii})\Delta t \text{ for } j = i \end{array} \right. \end{aligned}$$

(Note: in this case you should express everything in terms of the random variables $X_0, X_1, X_2, \dots, T_1, T_2, \dots$, and use their joint distribution to compute the desired probability)

PROBLEM SET 8 (DUE THURSDAY, NOVEMBER 10)

Exercise 36. Let X_t and Y_t be two independent Poisson processes with rate parameters λ and μ respectively, measuring the number of customers arriving in shops 1 and 2, respectively.

(a) What is the probability that a customer arrives in store 1 before any customer arrives in store 2?
(b) Show that $Z_t = X_t + Y_t$ is a Poisson process, and compute its rate.
(c) What is the probability that in the first four hours a total of 4 customers have arrived in the two stores?
(d) Given that exactly 4 customers have arrived at the two stores, what is the probability that the all went to store 1?
(e) Let $T_1^{(2)}$ be the time of arrival of the first customer in store 2. Then $X_{T_1^{(2)}}$ is the number of customers in store 1 at the time the first customer arrives in store 2. Find the probability distribution of the random variable $X_{T_1^{(2)}}$.
(f) Let T_1 be the first time that at least one customer has arrive in each of the two shops. Find the probability density function for T_1.

Exercise 37. Consider the continuous time Markov chain with state space $S = \{1, 2, 3, 4\}$ and infinitesimal generator

$$A = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 0 & -3 & 2 & 1 \\ 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

(a) Find the equilibrium distribution $\bar{\pi}$.
(b) Suppose the chain starts in state 1. What is the expected amount of time until it changes state for the first time?
(c) Again assume the chain starts in state 1. What is the expected amount of time until the chain reaches state 4?

Exercise 38. Consider the birth and death process with $\lambda_n = 1/(n+1)$ and $\mu_n = 1$. Show that the process is recurrent and admits an equilibrium distribution. Find the equilibrium distribution.

Exercise 39 (K&T 1.4 p.342)**.** A new product (a "Home Helicopter" to solve the commuting problem) is being introduced. The sales are expected to be determined by both media advertising and word-of-mouth advertising. Assume that media advertising creates new customers according to a Poisson process of rate $\alpha = 1$ customer per month. For the word-of-mpunth advertising, assume that each purchaser of a Home Helicopter will generate new customers at a rate $\theta = 2$ customers per month. Let X_t be the total number of Home Helicopter customers up to time t.

(a) Model X_t as a pure birth process by specifying the birth parameters λ_n, $n \neq 0$.
(b) What is the probability that exactly two helicopters are sold during the first month?

Exercise 40 (K&T 1.10 p.344)**.** Consider a pure birth process on the states $0, 1, \dots, N$, with birth parameters $\lambda_n = (N-n)\lambda$, $n = 0, 1, \dots, N$. Suppose that $X_0 = 0$. Determine $P_n(t) = \mathbb{P}[X_t = n]$ for $n = 0, 1, 2$.

PROBLEM SET 9 (DUE THURSDAY, NOVEMBER 17)

Exercise 41. Consider the Birth and Death process describing a population with immigration, with $\lambda_n = n\lambda + \nu$, $\mu_n = n\mu$.

(a) Fine the values of λ, μ, ν for which the chain is reducible/irreducible, recurrent/transient, and for which it admits an equilibrium distribution.
(b) If $\nu = 0$ (no immigration), find the values of λ and μ for which extinction is certain.

Exercise 42. Consider the MMk model describing a queue with k servers: $\lambda_n = \lambda$, $\mu_n = n\mu$ for $n \leq k$ and $\mu_n = k\mu$ for $n \geq k$.

(a) For $k = 1$, find the equilibrium distribution and the expected length of the queue.
(b) For $k = 2$, find the equilibrium distribution and the expected length of the queue.
(c) For $k = \infty$, find the equilibrium distribution and the expected length of the queue.

Exercise 43. Let X_1, X_2, $X_3, \dots$ be a sequence of independent, identically distributed random variables with mean μ and variance σ^2. Let $S_n = X_1 + \cdots + X_n = \sum_{n=1}^{n} X_n$, $n \geq 0$ be the partial sums.

(a) For every $m, n \geq 0$ compute the conditional expectation $\mathbb{E}[S_n \,|\, \mathcal{F}_m] = \mathbb{E}[S_n \,|\, X_1, \dots, X_m]$.
(b) For every $m, n \geq 0$ compute the conditional expectation $\mathbb{E}[S_n^2 \,|\, \mathcal{F}_m]$.

Exercise 44. A game is as follows: at each time n we toss a fair coin $X_n = \pm 1$ and we bet on $X_n = +1$ an amount B_n of dollars, which we decide every time based on the knowledge of the game up to time n. So, if $X_1 = 1, X_2 = 1$and $X_3 = -1$, at time 3 we will have won/lost the amount $B_1 +_2 - B_3$.

We formalize this as follows. Let X_1, X_2, $X_3, \dots$ be a sequence of independent, identically distributed Bernoulli random variables on $\{+1, -1\}$ with parameter p. And let B_n be random variable depending only on the variables $X_1, X_2, \dots, X_n$ (i.e. $\mathcal{F}_{n-1}$-measurable). The winnings/losses after n steps will be: $W_n = B_1 X_1 + B_1 X_2 + \cdots + B_n X_n = \sum_{i=1}^{n} B_i X_i$.

(a) Compute, for $n \geq 0$, $\mathbb{E}[W_{n+1} \,|\, \mathcal{F}_n]$.
(b) Deduce that $\mathbb{E}[W_{n+1} \,|\, \mathcal{F}_n] = W_n$ for every $n \geq 0$ (i.e. W_n describes a "fair game", or is a martingale) if and only if $p = 1/2$ (fair coin) or $B_n = 0$ for every n (no game).

Exercise 45. Consider an urn with balls of two colors, green and red. Assume that initially there is only one green and one red ball. At each time step a ball is chosen at random from the urn. If a green ball is chosen, it is returned and in addition another green ball is added to the urn. Similarly, if a red ball is chosen, it is returned and in addition another red ball is added to the urn. Let M_n be the fraction of green balls in the urn after n draws.

(a) Describe M_n as a (time inhomogeneous) Markov chain, and provide the transition probabilities at time n: $\mathbb{P}[M_{n+1} = y \,|\, M_n = x]$.
(b) Prove that M_n is a martingale, i.e. $\mathbb{E}[M_{n+1} \,|, \mathcal{F}_n] = M_n$.

PROBLEM SET 10 (DUE TUESDAY, NOVEMBER 29)

Exercise 46. Recall that in the Opional Sampling Theorem we assumed: (0) $\mathbb{P}[T < \infty] = 1$, (2) $\mathbb{E}[|M_T| \mathbb{1}_{\{T > N\}}] \overset{N \to \infty}{\longrightarrow} 0$, and (3) $\mathbb{E}[|M_N| \mathbb{1}_{\{T > N\}}] \overset{N \to \infty}{\longrightarrow} 0$. We want to show that condition (2) is very "weak". In particular, we want to prove that:

Claim: If $\mathbb{P}[T < \infty] = 1$ and $\mathbb{E}[|M_T|] < \infty$, then $\mathbb{E}[|M_T| \mathbb{1}_{\{T > N\}}] \overset{N \to \infty}{\longrightarrow} 0$.

(a) Let X be a random variable such that $\mathbb{E}[|X|] < \infty$. Prove that $\mathbb{E}[|X| \mathbb{1}_{\{X > N\}}] \overset{N \to \infty}{\longrightarrow} 0$.
(b) If A is an event such that $\mathbb{P}[A] \leq \mathbb{P}[\{|X| > N\}]$, then $\mathbb{E}[|X| \mathbb{1}_A] \leq \mathbb{E}[|X| \mathbb{1}_{\{|X| > N\}}]$.
(c) Let X be a random variable such that $\mathbb{E}[|X|] < \infty$, and let A_N be a sequence of events such that $\mathbb{P}[A_N] \overset{N \to \infty}{\longrightarrow} 0$. Prove that $\mathbb{E}[|X| \mathbb{1}_{A_N}] \overset{N \to \infty}{\longrightarrow} 0$.
(d) Apply the above conclusion to $X = M_T$ and $A_N = \{T > N\}$, to deduce the Claim.

Exercise 47. Let $X_0, X_1, X_2, \dots$ be a simple ($p = 1/2$) random walk starting at $X_0 = a$. Let $T = \inf\{n \,|\, X_n = 0 \text{ or } N\}$. We want to compute $\mathbb{E}[T]$.

(a) Prove that the random variables $M_n = X_n^2 - n$, $n \geq 0$ form a martingale.
(b) Prove that all the assumptions of the Optional Sampling Theorem hold.
(c) Use the Optional Sampling Theorem to compute $\mathbb{E}[T]$.

Exercise 48. Consider "Polya's urn" from execise 45: $M_0 = \frac{1}{2}$ and, given $M_n = \frac{a}{n+2}$, we have

$$M_{n+1} = \begin{cases} \frac{a+1}{n+3} & \text{with prob. } \frac{a}{n+2}, \\ \frac{a}{n+3} & \text{with prob. } \frac{n+2-a}{n+2}. \end{cases}$$

Recall from exercise 45 that M_n is a martingale, with values in $[0, 1]$.

By the Martingale Convergence Theorem we know that there exists a random variable M_∞ such that $\lim_{n \to \infty} M_n = M_\infty$. This exercise shows that M_∞ is the uniform random variable in $[0, 1]$.

Prove the following:

Claim: $\mathbb{P}[M_n = \frac{a}{n+2}] = \frac{1}{n+1}$ for every $a = 1, \dots, n+1$.

Exercise 49. Let $X_0, X_1, X_2, \dots$ be a non-symmetric random walk starting at $X_0 = a$, with absorbing barriers at 0 and N, with probability $p < 1/2$ of moving to the right, and probability $1 - p$ of moving to the left. Let T be the first time that the walk reaches 0 or N: $T = \inf\{n \,|\, X_n = 0 \text{ or } N\}$. In this exercise we want to compute: (1) the probability that the walk is "absorbed" at 0, $p_0 = \mathbb{P}[X_T = 0]$, and at N, $p_N = 1 - p_0$; (2) the expected time of absorption: $\mathbb{E}[T]$.

(a) Show that $M_n = \left(\frac{1-p}{p}\right)^{X_n}$ is a martingale.
(b) Show that all the assumptions of the Optional Sampling Theorem hold for M_n
(c) Use the Optional Sampling Theorem to compute p_0.
(d) Show that $W_n = X_n + (1-2p)n$ is a Martingale.
(e) Show that all the assumptions of the Optional Sampling Theorem hold for W_n
(f) Use the Optional Sampling Theorem to compute $\mathbb{E}[T]$.

Exercise 50. Let X_t, $t \geq 0$ be an irreducible Birth/Death process with parameters $\lambda_n > 0$ for every $n \geq 0$ and $\mu_n > 0$ for every $n \geq 1$. Let $T_1, T_2, \ldots$ be the jumping time intervals, and $X_1 = X_{T_1}$, $X_2 = X_{T_1+T_2}, \ldots$ be the states of jumps. In class we stated, without proof, the following result:
Theorem: $\{X_t\}_{t\geq 0}$ (or equivalently $\{X_n\}_{n\geq 0}$) is recurrent if and only if the only solution of the following system

$$\begin{aligned} &(\lambda_n + \mu_n)a(n) = \lambda_n a(n+1) + \mu_n a(n-1)\ , \quad n \geq 0\,, \\ &a(0) = 0\ , \quad 0 \leq a(n) \leq 1\ \forall n \geq 0\,, \end{aligned} \tag{2}$$

is $a(n) = 1$.

(a) Let $N = \inf\{n \geq 0 \,|\, X_n = 0\}$ and $a(n) = \mathbb{P}[N < \infty \,|\, X_0 = n]$. Let $N_n = \min\{n, N\}$. And let $M_n = a(X_{N_n})$, where $\{a(n)\}_{n\geq 0}$ is some solution of the system (2). Prove that M_n is a martingale.
(b) Prove that M_n satisfies all assumptions of the Martingale Convergence Theorem.
(c) Use the above fact to prove that Theorem (*Hint:* use the identity $\mathbb{E}[M_\infty] = \mathbb{E}[M_0]$).

PROBLEM SET 11 (DUE TUESDAY, DECEMBER 6)

Exercise 51 (K&T 1.2 p.488)**.** Let B_t, $t \geq 0$, be the standard Brownian motion. Compute $\mathbb{E}[e^{\lambda B(t)}]$ for every $\lambda \in \mathbb{R}$.

Exercise 52. Let B_t, $t \geq 0$, be the standard Brownian motion. Compute $\mathbb{P}[B_2 > 0 \,|\, B_1 > 0$. Are the events $\{B_2 > 0\}$ and $\{B_1 > 0\}$ independent?

Exercise 53. Let B_t, $t \geq 0$, be the standard Brownian motion. In this exercise we want to prove the following
Claim: $\lim_{t\to\infty} \frac{B_t}{t} = 0$ with probability 1.

(a) Notice that $B_1 - B_0$, $B_2 - B_1$, $\dots$, $B_n - B_{n-1}$ are independent identically distributed normal random variables of mean 0 and variance 1, and use the Law of Large Numbers to deduce that:

$$\lim_{n\to\infty} \frac{B_n}{n} = 0 \text{ with probability } 1\,.$$

(b) For $n \geq 0$ consider the random variable $M_n = \sup_{n \leq t < n+1}(|B_t - B_n|)$. Us the Reflection Principle (and an appropriate estimate of the resulting integral) to prove that

$$\mathbb{P}[M_n \geq a] = 4\mathbb{P}[B_1 \geq a] \leq \frac{8}{\sqrt{2\pi}a} e^{-a^2/2}\,.$$

(c) Prove that the expected number of times that $M_n \geq 2\sqrt{\log(n)}$ is:

$$\mathbb{E}\Big[\sum_{n=0}^{\infty} \mathbb{I}_{\{M_n \geq 2\sqrt{\log(n)}\}}\Big] < \infty\,.$$

(d) Deduce that, with probability 1, $M_n \geq 2\sqrt{\log(n)}$ only a finite number of times.

(e) Deduce that

$$\lim_{n\to\infty} \frac{M_n}{n} = 0 \text{ with probability } 1\,.$$

(f) Use (a) and (e) to deduce the Claim: $\lim_{t\to\infty} \frac{B_t}{t} = 0$ with probability 1.

Exercise 54. A collection of random variables $Y_1, \dots, Y_n$ are said to have a *joint normal distribution* with mean 0 if there exist independent independent identically distributed normal random variables $X_1, \dots, X_n$ of mean 0 and variance 1 such that

$$\begin{aligned} Y_1 &= a_{11}X_1 + \cdots + a_{1n}X_n\,, \\ &\vdots \\ Y_n &= a_{n1}X_1 + \cdots + a_{nn}X_n\,, \end{aligned}$$

for some constants $a_{ij} \in \mathbb{R}$, $i, j = 1, \dots, n$. In matrix form: $Y = AX$, where $Y = (Y_i)_{i=1}^n$, $X = (X_j)_{j=1}^n$, $A = \big(A_{ij}\big)_{i,j=1}^n$. The *covariance matrix* $\Gamma = \big(\Gamma_{ij}\big)_{i,j=1}^n$ for joint normal random variables $Y_1, \dots, Y_n$ is defines by:

$$\Gamma_{ij} = \mathbb{E}[Y_i Y_j]\,,\ i, j = 1, \dots, n\,.$$

(a) Express the covariant matrix Γ in terms of the matrix of coefficients A, for joint normal random variables $Y_1, \dots, Y_n$.

(b) Let B_t, $t \geq 0$, be the standard Brownian motion. Show that, for $t_1 \leq t_2 \leq \cdots \leq t_n$, the random variables $B_{t_1}, \dots, B_{t_n}$ have a joint normal distribution.

(c) Compute the covariance matrix Γ for the random variables $B_{t_1}, \dots, B_{t_n}$.

Exercise 55. In this exercise we show that the standard Brownian motion B_t is is *invariant* under appropriate "rescaling of space and time", and under appropriate "time inversion".

(a) Show that, for every $a > 0$, the stochastic process $X_t = \frac{1}{\sqrt{a}} B_{at}$ is a standard Brownian motion.

(b) Show that $Y_t = tX_{\frac{1}{t}}$ is a standard Brownian motion. (*Note:* by Exercise 53 we know that $Y_0 = 0$ with probability 1).