18.395 Group Theory with Applications to Physics
Problem Set 6 Due: Thursday, December 6, 2012

26. The Cartan-Weyl basis of a compact simple Lie algebra can be viewed as a rearrangement of the standard Hermitian basis $\{Y_i\}$ in which it is convenient to deal with root generators E_α which are non-Hermitian. Moreover, the C-W basis is set up to be consistent with the standard diagonal C-K metric. In the basis $\{Y_i\}$ we can write

$$\begin{aligned}\langle Y_i|Y_j\rangle &= \frac{1}{c}\,\mathrm{Tr}\,(\,\mathrm{ad}\,Y_i\,\mathrm{ad}\,Y_j) \\ &= \frac{1}{c}\sum_k \langle Y_k|\,\mathrm{ad}\,Y_i\,\mathrm{ad}\,Y_j|Y_k\rangle \\ &= \frac{1}{c}\sum_k \langle Y_k|Y_i,[Y_j,Y_k]\rangle \\ &= \frac{1}{c}\sum_{k,\ell,m} f_{jk\ell}f_{i\ell m}\langle Y_k|Y_m\rangle\end{aligned}$$

Then if we assume $\langle Y_k|Y_m\rangle = \delta_{km}$, we find that

$$\mathrm{Tr}\ \mathrm{ad}\,Y_i\,\mathrm{ad}\,Y_j = \sum_{k,\ell} f_{ik\ell}f_{jk\ell} = c\delta_{ij}\,.$$

This exercise does not prove anything, but it does show that things are consistent. It is interesting to perform similar checks of the consistency in the C-W basis where we wrote

$$\begin{aligned}\langle H_i|H_j\rangle &= \frac{1}{c}\,\mathrm{Tr}\ \mathrm{ad}\,H_i\,\mathrm{ad}\,H_j = \delta_{ij} \\ \langle H_i|E_\alpha\rangle &= \frac{1}{c}\,\mathrm{Tr}\ \mathrm{ad}\,H_i\,\mathrm{ad}\,E_\alpha = 0 \\ \langle E_\alpha|E_\beta\rangle &= \frac{1}{c}\,\mathrm{Tr}\ \mathrm{ad}\,E_{-\alpha}\,\mathrm{ad}\,E_\beta = \delta_{\alpha\beta}\end{aligned}$$

For example, we can write the second line as

$$\langle H_i|E_\alpha\rangle = \frac{1}{c}\left\{\sum_k\langle H_k|\,[H_i,[E_\alpha,H_k]]\rangle + \sum_{\beta\neq-\alpha}\langle E_\beta|\,[H_i,[e_\alpha,E_\beta]]\rangle + \langle E_{-\alpha}|\,[H_i,[E_\alpha,E_{-\alpha}]]\rangle\right\}.$$

Using the commutation relations of the C-W basis, we find in two stages that

$$\langle H_i|E_\alpha\rangle = -\frac{1}{c}\left\{\sum_k \alpha_k\alpha_i\langle H_k|E_\alpha\rangle + \sum_{\beta\neq-\alpha} N_{\alpha,\beta}\,(\alpha_i+\beta_i)\,\langle E_\beta|E_{\alpha+\beta}\rangle\right\}$$

Since all scalar products which appear are zero, we have only a trivial consistency check here. Things are less trivial in the other sectors and I ask you to show this as follows:

a) Calculate $\langle H_i|H_j\rangle$ as above by tracing over the basis $\{|H_k\rangle, |E_\alpha\rangle\}$ and show that consistency requires the geometrical relation among root vectors

$$\sum_\alpha \alpha_i\alpha_j = c\delta_{ij}$$

where the sum is over all non-zero roots of the algebra.

b) Show that $\langle E_\alpha | E_\beta \rangle = 0$ for $\alpha \neq \beta$ is rather trivially consistent.

c) Calculate $\langle E_\alpha | E_\alpha \rangle$ and deduce the consistency requirement

$$2\alpha \cdot \alpha + \sum_{\beta \neq -\alpha} |N_{\alpha,\beta}|^2 = c\,.$$

27. Suppose that α and β are roots of a compact, simple Lie algebra such that $[E_\alpha, E_\beta] = 0$. Show that $\alpha + \beta$ is not a root. In particular this means that for any root α,. 2α is not a root.

Hint: Consider the chains $E_{\beta+\alpha}, E_{\beta+2\alpha}, \ldots, E_{\beta+\alpha+p\alpha}$ and $E_\beta, E_{\beta-\alpha}, \ldots, E_{\beta-q\alpha}$ and apply the master formula.

28. Let $\sigma_a,\ \ a = 1,2,3$ and $\tau_i,\ \ i = 1,2,3$ be two sets of Pauli matrices which act in independent copies of C^2. The following list of 10 product matrices, formed from the $\sigma_a,\ \tau_i$ and the identity I. Products act in the tensor product $C^2 \otimes C^2$, but we do not indicate the $\otimes$ symbol explicitly:

$$\sigma_a\, I, \quad \sigma_a\, \tau_1, \quad \sigma_a\, \tau_3, \quad I\, \tau_2$$

These 10 matrices determine a simple algebra of rank two in which it is convenient to take $H_1 = \sigma_3$ and $H_2 = \sigma_3\, \tau_3$ as the Cartan subalgebra.

(a) Find the weights of the 4-dimensional representation generated by these matrices.

(b) Find the weights (roots!) of the adjoint representation.

(c) Find the simple roots and identify the positive roots as integer sums of the simple ones.

(d) What is the Dynkin diagram?