1 18.395 Group Theory and Physics PSet 5 Due: Thursday, November 15, 2012

21. The purpose of this problem is to provide a first acquaintance with the left-invariant 1-forms for the Lie groups $SU(n)$. Let θ_i be any set of $n^2 - 1$ parameters, and consider the differential of a general $n \times n$ special unitary matrix $A(\theta)_{\alpha\beta}$, namely

$$dA_{\alpha\beta} \equiv \sum_i \frac{\partial A_{\alpha\beta}}{\partial \theta_i} d\theta_i \ .$$

The left invariant 1-forms are defined by

$$\left(A^{-1} dA\right)_{\alpha\beta} = A^{-1}{}_{\alpha\gamma}\, dA_{\gamma\beta} \ .$$

From now on we will suppress matrix indices. See Tung, pp. 131–132, for a nice explanation of why $A^{-1}dA$ is invariant. Here we want to show that $A^{-1}dA$ is in the Lie algebra of $SU(n)$ which simply means that it is a traceless anti-Hermitean matrix.

a)Use the unitary condition $A^{+}A = \mathbb{1}$ to show that $\left(A^{-1}dA\right)^{+} = -A^{-1}dA$ which is the anti Hermitean property.

b)Use $\det A = 1$ to show that $\mathrm{Tr}\ A^{-1}dA = 0$. One instructive approach is to use $\det A = \exp(\mathrm{Tr}\ \ln A)$ so that $\mathrm{Tr}\ \ln A = 0$. Thus the deed will be done if you can demonstrate the following important general formula

$$d\,\mathrm{Tr}\ \ln A = \mathrm{Tr}\, A^{-1} dA \,,$$

which you can do using the class definition of $\ln A$ for any diagonalizable matrix. Another approach is to use formulas for $\det A$ and A^{-1} involving the alternating symbol $\varepsilon_{\alpha_1\alpha_2\cdots\alpha_n}$. This approach makes it clear that the desired property holds for any matrix function $A(x)$ for which $\det A(x) = \text{constant} \ \ (\neq 0)$.

Hint: The differential dA is a well defined $n \times n$ matrix, and one does not really need its "explicit" form as a linear combination of $d\theta_i$. This form was provided above as a crutch for the unsophisticated. You should be able to present the solution to parts a) and b) together on a single page.

22. This problem is designed to give you more practice in manipulations of Dirac γ matrices using the defining anti-commutator of the Clifford algebra. You may assume even spacetime dimension, although the results below hold both for even and odd. We are especially concerned with the fact that the second rank elements $\gamma^{\mu\nu}$ (multiplied by 1/2) act as generators of rotations.

a) Show that

$$[\gamma^{\mu\nu}, \gamma^{\rho}] \;=\; 2\,(\delta^{\nu\rho}\gamma^{\mu} - \delta^{\mu\rho}\gamma^{\nu})$$

b) Show that

$$[\gamma^{\mu\nu}\,,\,\gamma^{\rho\sigma}] \;=\; 2(\delta^{\nu\rho}\gamma^{\mu\sigma} - \delta^{\mu\rho}\gamma^{\nu\sigma} - \delta^{\nu\sigma}\gamma^{\mu\rho} + \delta^{\mu\sigma}\gamma^{\nu\rho})$$

This commutator is essentially the Euclidean version of Lorentz group generators of Problem 9c. See also Prob. 24 below. Hint: You can simplify the computation needed for part b) by finding a simple way to use the result of part a).

23. The group of scale/translations of the real line sends $x \to x^{'} = \alpha x + b$. You can find its Lie algebra from the group composition law discussed in class. However, it is more fun to recognize that the differential operators $X_1 = x\frac{d}{dx}$ and $X_2 = \frac{d}{dx}$ are a basis for the same Lie algebra. Show that the set of commutators of these operators are closed, i.e. that $[X_i, X_j] = f_{ij}{}^k X_k$. Obtain the structure constants and compute the Cartan-Killing form g_{ij} from them. Show that the $C-K$ form is degenerate because the Lie algebra is not semi-simple.

24. The purpose of this problem is to illustrate the features of the Cartan-Killing form and the quadratic Casimir operator using the Lie algebra $so(N)$ in the Hermitian basis in which

$$\left[Y^{ij}, Y^{k\ell}\right] = i\left(\delta^{jk}Y^{i\ell} - \delta^{ik}Y^{j\ell} + \delta^{i\ell}Y^{jk} - \delta^{j\ell}Y^{ik}\right) .$$

We would expect that the metric in any irrep. $Y^{ij} \to T^{ij}$ would take the form $\mathrm{Tr}\left(T^{ij}T^{k\ell}\right) = c_T\Delta^{ij;k\ell}$ where $\Delta^{ij;k\ell} = \left(\delta^{ik}\delta^{j\ell} - \delta^{i\ell}\delta^{jk}\right)$ is the analogue of the Kronecker delta in the space of anti-symmetric tensors.

Note that $\frac{1}{2}\Delta^{ij;mn}\Delta^{mn;k\ell} = \Delta^{ij;k\ell}$. The quadratic Casimir operator is $C^{(2)} = \frac{1}{2C}T^{ij}T^{ij}$, and in any irreducible representation $C^{(2)} = {\lambda_T}^{(2)}\mathbb{1}$. The relation between γ_T and ${\lambda_T}^{(2)}$ derived in class is

$$\gamma_T = \frac{2d_T}{N(N-1)}{\lambda_T}^{(2)}$$

where d_T is the dimension of the irrep.

a) For the adjoint rep show explicitly that

$$\mathrm{Tr}\,\left(\mathrm{ad}\,Y^{ij}\,\mathrm{ad}\,Y^{k\ell}\right) = 2(N-2)\Delta^{ij;k\ell} \equiv c\Delta^{ij;k\ell}\,.$$

Show that $C^{(2)}$ is proportional to $\mathbb{1}$ with ${\lambda_T}^{(2)} = 1$ as required. Hint: use

$$\mathrm{ad}\,Y^{ij}\,\mathrm{ad}\,Y^{k\ell}\,Y^{mn} = \left[Y^{ij}, \left[Y^{k\ell}, Y^{mn}\right]\right]$$

and then take the trace by computing

$$\frac{1}{2}\left\langle Y^{mn}\right|\left[Y^{ij}\left[Y^{k\ell}, Y^{mn}\right]\right]\rangle \qquad \text{with} \qquad \langle Y^{pq}|Y^{rs}\rangle = \Delta^{pq;rs}\,.$$

b) Compute $\mathrm{Tr}\left(T^{ij}T^{k\ell}\right)$ and $C^{(2)}$ for the vector representation, using $D\left(T^{ij}\right)_{ab} = iM^{(ij)}{}_{ab}$ which comes from Problem 9b). Verify the relation between γ_T and $\lambda_T{}^{(2)}$.

25. Show that the 2-dimensional defining representation of the group $SU(2)$ is pseudoreal, specifically that it is equivalent to its complex conjugate, but inequivalent to a representation by real matrices.