18.395 **Group Theory with Applications to Physics**

Problem Set #4 **Due: Thursday, November 1, 2012**

16. This problem relies on results of Problem 9, and is designed to explore irreducible representations on a real carrier space and the weaker implications of Schur's Lemmas. Refer to Problem 9d and consider the explicit real 4×4 matrices $I_i^+ = \frac{1}{2}(I_i + K_i)$, for $i = 1, 2, 3$. Your work on that problem showed that the commutators of these matrices were those of the Lie algebra of $SU(2)$. Thus the I_i^+ can be considered as giving a 4 dimensional real representation of this Lie algebra. The term *real representation* means here that the carrier space is a real vector space, and a real representation of $SU(2)$ in any set of 3 real matrices M_i which satisfy $[M_i, M_j] = \varepsilon_{ijk} M_k$.

Show that the 4×4 matrices I_i^+ are an irreducible representation of the Lie algebra. Then simply observe that the 3 other matrices $I_j^- = \frac{1}{2}(I_j - K_j)$ commute with the I_i^+ as was already shown in Problem 7. The I_j^- are certainly not multiples of the unit matrix, and one can show that this is a case where the set of matrices which commute with the I_i^+ is isomorphic to the quaternions.

Hint: The hard part of this problem is to find a test for the presence of invariant subspaces which can be applied concretely to show that there are no invariant subspaces. I was able to formulate a test which reduced to a computation which took less than 2 pages. Perhaps students can find a simpler method. However, they should use their heads rather than get involved in long tedious computations. Believe it or not, this problem is related to the theory of instantons in non-Abelian gauge theories.

17. This is a problem on the use of character methods for group representations to ascertain the decomposition of direct product representations of $SU(2)$ into irreducible components.

- **a)** In class we considered the direct product of two 3-dimensional representations with supermatrix $D(R)_{ij;k\ell} = R_{ik}R_{j\ell}$ and we asserted that it was the direct sum i.e. $1 \otimes 1 = 2 \oplus 1 \oplus 0$ in an angular momuntum notation where j denotes the $2j+1$ dimensional irrep. Prove this by computing the compound character and showing that it is the appropriate superposition of simple characters.
- **b)** Find the decomposition of $1 \otimes 1 \otimes \frac{1}{2}$ by expressing the compound character as an integer superposition of simple characters.

Hint: In the irrep with angular momemtum j, the eigenvalues of a rotation by angle θ about any axis are $exp(-im\theta)$ with $m = j, j-1, \ldots, -j$.

18. The *commutator subgroup* G_c is the subgroup of G generated by all elements of the form $ghg^{-1}h^{-1}$ where g, h are elements of G. Prove that G_c is a normal subgroup of G and that G/G_c is commutative.

19. For any finite group G the commutator subgroup G_c is an invariant subgroup and G/G_c is commutative. Use this to show that the number of inequivalent one-dimensional irreps of G is equal to $n(G)/n(G_c)$. The problem can be divided into two parts:

- **a)** Show that there is a $1:1$ correspondence between irreps of G/G_c and one-dimensional irreps of G.
- **b)** Argue simply that an abelian group of order n has n inequivalent one-dimensional irreps.

See Problem 20 on next page.

20. For the Clifford algebra of a D dimensional Euclidean space, use only the defining anti-commutator to show that

$$\gamma_\nu \gamma^\mu \gamma^\nu = (2 - D)\gamma^\nu$$
$$\gamma_\rho \gamma^{\mu\nu} \gamma^\rho = (D - 4)\gamma^{\mu\nu}$$

where repeated indices are summed. Formally the metric tensor $\delta^{\mu\nu}$ is used to raise and lower indices. Calculate $\mathrm{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)$ in terms of the metric tensor and the dimension N of the matrices.