18.395 Group Theory with Applications to Physics
Problem Set #3 Due: Thursday, October 18, 2012

11 The first two problems of this PSet deal with the group D_3 of symmetries of the equilateral triangle. Please use the notation and the multiplication table given in an early class lecture.

The group has 4 proper subgroups which we list as follows:
$H_A = (E, A), \quad H_B = (E, B), \quad H_C = (E, C), \quad H_R = (E, D, F).$

11a) Show that the first three groups are conjugate to one another. I.e. find group elements such that
$H_B = gH_A\, g^{-1}, \quad H_C = g' H_A\, g'^{-1}, H_C = g\text{"}\, H_B$
This means that these 3 subgroups are not invariant.

11b) Write down the left coset decomposition of the quotient space D_3/H_A in the form
$D_3/H_A = (E, A) \cup (\;,\;) \cup (\;,\;).$
Each (,) contains two elements of the group which you must find. Show that the coset multiplication law that works for an invariant subgroup fails to be consistent in this case.

11c) Show that H_R is an invariant subgroup and write down the coset decomposition of the quotient group D_3/H_R in analogous form.

12a) Show that the following is a representation:

$$D(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad D(A) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad D(C) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$D(D) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad D(F) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

12b) Show that the representation is reducible by determining an invariant subspace of the carrier space. Show that it is completely reducible by finding a complementary subspace which is invariant. (Some intelligent guesswork may be involved.) Def: Given a subspace $V^p \subset V^n$, then a complementary subspace is any V^q, with $p + q = n$ with the properties i) $V^p \cap V^q = 0$ and ii) any $v \,\epsilon\, V^n$ can be expressed as $v = u + w$ with $u \,\epsilon\, V^p$ and $w \,\epsilon\, V^q$.

12c) The group of the triangle is clearly a subgroup of the orthogonal group $O(2)$. Use this observation to construct a two- dimensional representation of the group.

13. The purpose of this problem is to explore the formula $\det A = \exp \mathrm{Tr} \ln A$ under broader conditions than discussed in class. Consider the case where $A^\alpha{}_\beta$ is a $GL(n, C)$ matrix which differs infinitesimally from the unit matrix, i.e., $A = \mathbf{1} + \epsilon M$ where ϵ is a small parameter.

(a) In class we derived the first few terms of the power series expansion

$$\det A = 1 + \epsilon \mathrm{Tr} M + \frac{1}{2}\epsilon^2 \left[(\mathrm{Tr} M)^2 - \mathrm{Tr}(M^2) \right] + \cdots$$

Verify this expression directly using the definition

$$\epsilon_{\alpha_1 \alpha_2 \cdots \alpha_n} A^{\alpha_1}{}_{\beta_1} A^{\alpha_2}{}_{\beta_2} \cdots A^{\alpha_n}{}_{\beta_n} = \det A \epsilon_{\beta_1 \beta_2 \cdots \beta_n}$$

where $\epsilon_{\alpha_1 \cdots \alpha_n}$ is the n-dimensional alternating symbol. You may need to use the standard results for contractions such as $\epsilon_{\alpha \beta_2 \cdots \beta_n} \epsilon^{\beta \beta_1 \cdots \beta_n}$. Figuring these out is part of the problem, but you need not write about how you figured them out.

(b) Find explicitly the order ϵ^3 term of the series.

14.Let ϕ be a homomorphism from a group G into a group G' with image $\phi(G)$ and kernel $\ker \phi$. Show that the quotient group $G/\ker \phi$ is isomorphic to the image $\phi(G)$. You may use the results of previous homework Problem 4.

15.Consider the direct product $D(g) \times D^*(g)$ of a representation with its conjugate. Show that this is reducible if $D(g)$ is unitary and the dimension of $D(g)$ is greater than one. (Hint: Show that the carrier space has an invariant subspace.)