18.395 Group Theory with Applications to Physics

Problem Set #2 Due: Thursday, October 4, 2012

3. Do the problem on Euler angles from PSet 1. Given the vector $\vec{v} = (\theta, 0, 0)$ with $-\pi < \theta < \pi$, express the rotation $R = e^{\vec{v}\cdot\vec{I}}$ in standard Euler form.

7. The Lorentz group of special relativity is the group of matrices $\Lambda^\mu{}_\nu$ which preserves the Minkowski metric tensor. This property is expressed by $\Lambda^\mu{}_\rho g_{\mu\nu} \Lambda^\nu{}_\sigma = g_{\rho\sigma}$. A contravariant vector v^μ transforms as $v^\mu \to v'^\mu = \Lambda^\mu{}_\nu v^\nu$ and its length $v^\mu g_{\mu\nu} v^\nu$ is invariant. It is frequently convenient to raise and lower indices using the metric tensor.

- **7a)** Use the information above to show that $\Lambda_\mu{}^\nu \equiv g_{\mu\rho} g^{\nu\sigma} \Lambda^\rho{}_\sigma = (\Lambda^{-1})^\nu{}_\mu$.
- **7b)** Show that the transformation law of the covariant vector $v_\mu = g_{\mu\rho} v^\rho$ can be written in the two equivalent ways $v_\mu \to \Lambda_\mu{}^\nu v_\nu = v_\nu (\Lambda^{-1})^\nu{}_\mu$.

8 The group of the triangle is clearly a subgroup of the orthogonal group $O(2)$. Use this observation to construct a two dimensional representation of the group.

9. Consider a pseudo-Euclidean metric $g_{\mu\nu} = (+\ldots+, -\ldots-)$ of signature (p, q) and the group $O(p, q)$ of transformations $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ which preserve the length of $x^\mu g_{\mu\nu} x^\nu$. Consider the elements of $O(p, q)$ which are infinitesimally close to the identity, i.e., $\Lambda^\mu{}_\nu$ of the form

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \varepsilon M^\mu{}_\nu$$

which preserves the length to first order in the small parameter ε. The matrix $M^\mu{}_\nu$ is called the generator of $SO(p, q)$. These generators determine a real vector space, since if $M^\mu{}_\nu$ and $M'^\mu{}_\nu$ are generators so is $aM^\mu{}_\nu + bM'^\mu{}_\nu$ where a and b are real constants. This vector space is the Lie algebra of $SO(p, q)$.

- **a)** Show that the vector space has dimension $\frac{1}{2}n(n-1)$ where $n = p + q$.
- **b)** Show that a basis of the vector space is given by the matrices $M^{(\alpha\beta)\mu}{}_\nu = g^{\alpha\mu}\delta^\beta_\nu - \delta^\alpha{}_\nu g^{\beta\mu}$ where the generator $M^{(\alpha\beta)} = -M^{(\beta\alpha)}$ generates transformations in the $(\alpha\beta)$ plane. One way to show that you have a basis is to prove orthogonality under the trace norm $(M, M') \equiv Tr(MM')$.
- **c)** Demonstrate the commutation relations:

$$\left[M^{(\alpha\beta)}, M^{(\gamma\delta)}\right] = g^{\beta\gamma} M^{(\alpha\delta)} - g^{\alpha\gamma} M^{(\beta\delta)} - g^{\beta\delta} M^{(\alpha\gamma)} + g^{\alpha\delta} M^{(\beta\gamma)} \; .$$

- **d)** For the case of $SO(4)$, use the notation $I_i = -\frac{1}{2}\varepsilon_{ijk} M^{(jk)}$ and $K_i = M^{(i4)}$ where $i, j, k = 1, 2, 3$. Write the commutators $[I_i, I_j]$, $[I_i, K_j]$ and $[K_i, K_j]$ using the ε_{ijk} symbol. Define the linear combinations $I_i^\pm = \frac{1}{2}(I_i \pm K_i)$. Calculate the commutators $\left[I_i^\pm, I_j^\pm\right]$ and $\left[I_i^\pm, I_j^\mp\right]$ and interpret the results.

10. In class, we first showed abstractly that the map $x \to x' = A \times A^\dagger$ gave a homomorphism, $\Lambda = \phi(A)$ from $SL(2, C)$ into the Lorentz group. Later we showed concretely that $\Lambda^\mu{}_\nu = \frac{1}{2}\mathrm{Tr}\left(\overline{\sigma}^\mu A \sigma_\nu A^\dagger\right)$. If $\Lambda' = \phi(B)$ denotes a second transformation, then the homomorphism property implies that the formula

$$\begin{aligned}
\Lambda^\mu{}_\rho \Lambda'^\rho{}_\nu &= \frac{1}{4}\mathrm{Tr}\left(\overline{\sigma}^\mu A \sigma_\rho A^\dagger\right) \mathrm{Tr}\left(\overline{\sigma}^\rho B \sigma_\nu B^\dagger\right) \\
&= \frac{1}{2}\mathrm{Tr}\left(\overline{\sigma}^\mu A B \sigma_\nu B^\dagger A^\dagger\right)
\end{aligned}$$

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is correct. Demonstrate by explicit matrix arguments that the trace equality above is indeed correct. Possible hint: Show that for any pair of 2x2 matrices M, N it is true that

$$\mathrm{Tr}\,(M\sigma_\rho)\mathrm{Tr}(\overline{\sigma}^\rho N) = 2\mathrm{Tr}(MN).$$

yh:misc/dzf