18.395 Group Theory with Applications to Physics
Problem Set 1 Due: Thursday, September 20, 2012

1. Find the multiplication law for a group with three elements and prove that it is unique.

2. Let u_α and v_α denote vectors of C^2 and let $\varepsilon^{\alpha\beta}$ be the two-dimensional alternating symbol ($\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$ and $\varepsilon^{12} = 1$). Consider the symplectic scalar product on C^2, defined by $(u, v) = \varepsilon^{\alpha\beta} u_\alpha v_\beta$ where repeated indices are summed. In English, $(u, v) = u_1 v_2 - u_2 v_1$.

a. Show that the the subgroup of $GL(2, C)$ which preserves this scalar product is $SL(2, C)$.

b. Let w_α and y_α denote two additional vectors. Prove the Schouten identity for scalar products

$$(u, v)(w, y) + (v, w)(u, y) + (w, u)(v, y) = 0\,.$$

3. Given the vector $\vec{v} = (\theta, 0, 0)$ with $-\pi < \theta < \pi$, express the rotation $R = e^{\vec{v}\cdot\vec{I}}$ in standard Euler form.

4. Consider a homomorphism from a group G into another group G'. To each $g \varepsilon G$ there is a $g' = \phi(g)$ in G', such that $\phi(g_1)\phi(g_2) = \phi(g_1 g_2)$.

a.Prove that the image $\phi(G)$ of the entire group G is a subgroup of G'.

b.The kernel of the homomorphism, called $\ker\phi$, is defined as the set of all $g \varepsilon G$ such that $\phi(g) = e'$, the identity element in G'. Prove that $\ker\phi$ is a subgroup of G. Given any $g_0 \;\varepsilon\; \ker\phi$ and any $g \;\varepsilon\; G$, show that $g\, g_0\, g^{-1} \;\varepsilon\; \ker\phi$. (Remark: this shows that $\ker\phi$ is an invariant subgroup of G.)

5. Consider the group $E(3)$ of combined rotations and translations in 3-dimensional space. A group element is specified by a rotation matrix R and a translation parameter a and can be denoted by $g(R, a)$. The action of $g(R, a)$ on a point x in space is given by $x \to Rx + a$.

a. Write the group product $g(R, a)g(R', a')$ as a single transformation $g(R'', a'')$ where R'' and a'' are expressed in terms of R, R', a, and a'. Express $g^{-1}(R, a)$ in the standard form.

b. Construct a 4-dimensional faithful representation of $E(3)$.

6. Derive the Schouten identity for $d = 3$. For any 3-vector v_i show that

$$\epsilon_{ijk} v_l - \epsilon_{jkl} v_i + \epsilon_{kli} v_j - \epsilon_{lij} v_k = 0. \qquad (1)$$