1 Approximate Poincaré Map for the van der Pol equation

Statement: Approximate Poincaré Map for the van der Pol equation

Consider the van der Pol equation, written in the form

\[
\begin{aligned}
\dot{x} &= -y, \\
\dot{y} &= x + \mu (1 - x^2) y,
\end{aligned}
\] where \(0 < \mu \ll 1\). (1.1)

Construct an approximation (using the fact that \(\mu\) is small) to the Poincaré map defined below, and use it to show that the system has a unique, stable, limit cycle. Define the Poincaré map \(z \to u = P(z)\) as follows:

1. For every \(0 < z\) (but not too large), let \(x = X(t, z)\) and \(y = Y(t, z)\) be the solution of (1.1) defined by \(X(0, z) = z\) and \(Y(0, z) = 0\).

2. For this solution the polar angle \(\theta\) in the phase plane is an increasing function of time, starting at \(\theta = 0\) for \(t = 0\). Thus, there is a time \(t = t_z\) at which the solution reaches \(\theta = 2\pi\) (note that \(t_z\) is a function of \(z\)). Then take \(u = X(t_z, z)\). (1.2)
Further questions:

3. In item 1 we state that \( z \) cannot be too large. Why? After you are done, check where is this used.

4. In item 2 we state that the solution is an increasing function of time. Show this.

Hint(s). Because \( t_z \) is a function of \( z \), unknown a priori, the definition of the Poincaré map above is a bit awkward to implement. To avoid having to calculate \( t_z \) for each solution, proceed as follows:

(a) Because \( \theta \) is an increasing function of time along the solution of interest, it can be parameterized by \( \theta \) instead of time.

(b) Write the equations in polar coordinates, and from this form obtain \( F \) in the equation

\[
\frac{dr}{d\theta} = F(r, \theta). \tag{1.3}
\]

(c) Now the Poincaré map can be computed as follows: solve (1.3) with the initial condition \( r(0) = z \). Then \( u = r(2\pi) \).

(d) To solve (1.3) expand the solution using the fact that \( \mu \) is small.

Note that, because you are only interested in \( 0 \leq \theta \leq 2\pi \), ‘secular’ terms do not matter.

2 Bifurcations of a Critical Point for a 1-D map

Statement: Bifurcations of a Critical Point for a 1D map

When studying bifurcations of critical points in the lectures, we argued that we could understand most of the relevant possibilities simply by looking at 1-D flows. The argument was based on the idea that, generally, a stable critical point will become unstable in only one direction at a bifurcation — so that the flow will be trivial in all the other directions, and we need to concentrate only on what occurs in the unstable direction. The only important situation that is missed by this argument, is the case when two complex conjugate eigenvalues become unstable (Hopf bifurcation).

Because in real valued systems eigenvalues arise either in complex conjugate pairs or as single real ones, the cases where only one (or a complex pair) go unstable are the the only ones likely to occur (barring situations with special symmetries that “lock” eigenvalues into synchronous behavior).

The same argument can be made when studying bifurcations of limit cycles in any number of dimensions. In this case one considers the Poincaré map near the limit cycle,\(^1\) with the role of the eigenvalues taken over by the Floquet multipliers. Again, we argue that we can understand a good deal of what happens by replacing the (multi-dimensional) Poincaré map by a one dimensional map with a stable fixed point, and asking what can happen if the fixed point changes stability as some parameter is varied. In this problem you will be asked to do this.

Remark 2.1 Some important cases are missed by this approach: the case where a pair of complex Floquet multipliers becomes unstable (Hopf bifurcation of a limit cycle), and the cases where a bifurcation occurs because of an interaction of the limit cycle with some other object (e.g., a critical point). Several examples of these situations can be found in section 8.4 of Strogatz’ book (Global Bifurcations of Cycles).

Consider a one dimensional (smooth) map from the real line to itself

\[
x \rightarrow y = f(x, \mu) \tag{2.1}
\]

that depends on some (real valued) parameter \( \mu \). **Assume that \( x = 0 \) is a fixed point for all values of \( \mu \) — that is, \( f(0, \mu) \equiv 0 \).** Furthermore, assume that \( x = 0 \) is stable for \( \mu < 0 \) and unstable for \( \mu > 0 \). That is:

\[
\left| \frac{\partial f}{\partial x} (0, \mu) \right| < 1 \quad \text{for} \quad \mu < 0, \quad \text{and} \tag{2.2}
\]

\[
\left| \frac{\partial f}{\partial x} (0, \mu) \right| > 1 \quad \text{for} \quad \mu > 0. \tag{2.3}
\]

\(^1\) The limit cycle is a fixed point for this map.
A further assumption, that involves no loss of generality (since the parameter $\mu$ can always be re-defined to make it true) is that
\[ \frac{\partial^2 f}{\partial x \partial \mu}(0, 0) \neq 0. \] (2.4)

This guarantees that the loss of stability is linear in $\mu$, as $\mu$ crosses zero. This is what is called a transversality condition. It means this:

Graph of the Floquet multiplier $\frac{\partial f}{\partial x}(0, \mu)$ as a function of $\mu$. Then the resulting curve crosses one of the lines $y = \pm 1$ transversally (curves not tangent at the common point) for $\mu = 0$.

By doing an appropriate expansion of the map $f$ for $x$ and $\mu$ small (or by any other means), show that (generally\(^2\)) the following happens:

a. For $\frac{\partial f}{\partial x}(0, 0) = 1$, either:
   
   a1. Transcritical bifurcation (no special symmetries assumed for $f$): There exists another fixed point, $x_0 = x_0(\mu) = O(\mu)$, such that: $x_0 \neq 0$ is unstable for $\mu < 0$ and $x_0 \neq 0$ is stable for $\mu > 0$. The two points “collide” at $\mu = 0$ and exchange stability.
   
   a2. Supercritical or soft pitchfork bifurcation, assuming that $f$ is an odd function of $x$: Two stable fixed points exist for $\mu > 0$, one on each side of $x = 0$, at a distance $O(\sqrt{\mu})$. All three points merge for $\mu = 0$.
   
   a3. Subcritical or hard pitchfork bifurcation, assuming that $f$ is an odd function of $x$: Two unstable fixed points exist for $\mu < 0$, one on each side of $x = 0$, at a distance $O(\sqrt{-\mu})$. All three points merge for $\mu = 0$.

   What does all this mean in the context of the Poincaré map for a limit cycle?

b. For $\frac{\partial f}{\partial x}(0, 0) = -1$ (no special symmetries assumed for $f$), either:
   
   b1. Supercritical or soft flip bifurcation: For $\mu > 0$ two points $x_1(\mu) \approx -x_2(\mu) = O(\sqrt{\mu})$ exist, on each side of the fixed point $x = 0$, with $x_2 = f(x_1, \mu)$ and $x_1 = f(x_2, \mu)$. Thus $\{x_1, x_2\}$ is a period two orbit for the map (2.1). Show that this orbit is stable.
   
   b2. Subcritical or hard flip bifurcation: For $\mu < 0$ two points $x_1(\mu) \approx -x_2(\mu) = O(\sqrt{-\mu})$ exist, on each side of the fixed point $x = 0$, with $x_2 = f(x_1, \mu)$ and $x_1 = f(x_2, \mu)$. Thus $\{x_1, x_2\}$ is a period two orbit for the map (2.1). Show that this orbit is unstable.

In the context of the Poincaré map for a limit cycle, a flip bifurcation corresponds to a period doubling bifurcation of the limit cycle.

**Hint 2.1** If you expand $f$ in a Taylor expansion near $x = 0$ and $\mu = 0$, up to the leading order beyond the trivial first term ($f \sim \pm x$), and you make sure to keep all the relevant terms (and nothing else), and you make sure to identify certain terms that must vanish, the problem reduces to (mostly) trivial high-school algebra. In figuring out what to keep and what not to keep, note that you have two small quantities ($x$ and $\mu$), whose sizes are related. The process is very similar to the Hopf bifurcation expansion calculation (e.g. see course notes), but much simpler computationally.

For part (b) you will have to keep in the expansion not just the leading order terms beyond the trivial first term, but the next order as well. Reason: In this case, you will be looking for solutions to the equation $f(f(x, \mu), \mu) = x$. But, when you calculate $f(f(x, \mu), \mu)$, you will see that the second order terms cancel out — thus the need for an extra term in the expansion. This is the same phenomena that forces the Hopf bifurcation calculation to third order. ✴

**IMPORTANT**

To standardize the notation, use the following symbols in your answer:

---

\(^2\) There are special conditions under which all this fails. You must find them as part of your analysis. What are they?
\[ \nu = \frac{\partial f}{\partial x}(0, 0) \] — note that \( \nu = \pm 1, \)

\[ a = \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(0, 0), \quad b = \frac{\partial^2 f}{\partial x \partial \mu}(0, 0), \quad c = \frac{1}{6} \frac{\partial^3 f}{\partial x^3}(0, 0), \quad \text{and} \quad d = \frac{1}{2} \frac{\partial^3 f}{\partial x^2 \partial \mu}(0, 0). \]

Then obtain leading order expressions for the fixed points and flip-bifurcation orbits in terms of these quantities.

## 3 Coastline fractal

### Statement: Coastline fractal

In this problem we construct a fractal that is a very idealized caricature of what a coastline looks like. The construction proceeds by iteration of a basic process, which we describe next.

We start with a simple curve, \( \Gamma_0 \), and apply to it a process that yields a new curve \( \Gamma_1 \). This new curve is made up of several parts, each of which is a scaled down copy of \( \Gamma_0 \). The same process is then applied to each of these parts, yielding \( \Gamma_2 \). Iterating, we obtain a series of curves \( \Gamma_n \), for \( n = 0, 1, 2, 3 \ldots \). The fractal is the limit of this process: \( \Gamma = \lim_{n \to \infty} \Gamma_n \) — provided that the limit exists.

For the coastline fractal we start by picking an angle \( 0 < \Theta < \pi \), and a length \( R_0 > 0 \). The first curve is:

\[ \Gamma_0 = \text{circular arc of radius } R_0, \text{ subtending an angle } \Theta. \]  (3.1)

Next divide \( \Gamma_0 \) into three equal sub-arcs, each subtending an angle \( \Theta/3 \), and replace each of these pieces by a properly scaled version of \( \Gamma_0 \). This yields \( \Gamma_1 \). The process is then repeated on each of the three pieces making up \( \Gamma_1 \), so as to obtain \( \Gamma_2 \), and so on ad infinitum. The first two steps in this construction are illustrated in figure 3.1.

![Figure 3.1: This figure illustrates the first two steps in the construction of the coastline fractal. That is, the curve \( \Gamma_0 \) (solid blue) and the curve \( \Gamma_1 \) (solid black). The dashed red lines indicate various radial lines useful in the construction.](image)

Now we show that the limit \( \lim_{n \to \infty} \Gamma_n \) exists. Consider an arbitrary radial line within the circle sector associated with \( \Gamma_0 \), and the intersection of this line with \( \Gamma_n \). It should be clear that this intersection is unique. Let \( d_n \) be the distance of this intersection from the origin of the radial line. Then \( \{d_n\} \) is an increasing, bounded sequence — thus it has a limit. This limit defines a point along the radial line. The set of all these points is the fractal \( \Gamma \). With a little more work one can show as well that \( \Gamma \) is described via a continuous function \( r = R(\Theta) \).

**Now do the following:**

1. For each \( n = 0, 1, 2, 3 \ldots \), calculate \( \ell_n = \text{length of } \Gamma_n \). What do you conclude about the “length” of \( \Gamma \)?

2. Calculate the fractal dimension (self-similar or box) of \( \Gamma \).
Hint: The first thing that you need to calculate is the “scaling” factor between $\Gamma_0$ and each of the three parts that make up $\Gamma_1$. With this scaling factor, which depends on $\Theta - 0 < S_c = S_c(\Theta) < 1$, everything else follows.

Note: Real coastlines are not this simple, of course. At the very least the number of parts into which each sector is divided should not be a constant (3 here), nor should the parts be equal in size, nor should they all subtend the same angle $\Theta$. Further: the sectors need not be exactly circular — though, this probably is not a terrible approximation.

4 Floquet Theory #01

Statement: Floquet Theory #01

Consider the (linear) system of ode

\[ \dot{x} = A(t) x, \quad (4.1) \]

where $x$ is an $n$-vector and $A$ is an $n \times n$ matrix valued periodic function of time — let $T > 0$ be the period.

The system is called stable $\iff$ all its solutions are bounded as $t \to \infty$.

Stability can be decided by looking at the eigenvalues of the Floquet matrix $R$ — the Floquet multipliers.

The system is stable $\iff$ all the Floquet multipliers are in the unit disk.

The Floquet matrix can be defined as follows: let $U = U(t)$ be the matrix valued solution to (4.1) satisfying the initial data $U(0) =$ identity. Then $R = U(T)$. (4.2)

Obviously, stability is a property that should not change under a time shift. Thus, define an “alternative” Floquet matrix $Q$ via: let $V = V(t)$ be the matrix valued solution to (4.1) satisfying the initial data $V(t_0) =$ identity. Then $Q = V(t_0 + T)$, (4.3) where $t_0$ is some arbitrary time.

It should then be that $R$ and $Q$ give the same answer as far as stability.

Prove that $R$ and $Q$ are similar matrices. (4.5)

That is, there is a non-singular matrix $S$ such that $Q = S R S^{-1}$ (4.6)

Note that (4.5) is stronger than (4.4). It also ensures that the decay rates (or growth rates, when instability arises) are the same, etc., etc.

Hints:

(A) Recall that both $U$ and $V$ are nonsingular — since their determinants satisfy the equation $\dot{d} = \text{Tr}(A) d$.

(B) Use the uniqueness theorem for solutions to show that $U(t + T) = U(t) R$. (4.7)

(C) Similarly, show that $V(t + T) = V(t) Q$. (4.8)

(D) Again, use the uniqueness theorem for solutions to write $V$ in terms of $U$.

These things will give you the tools you need to prove (4.5).

5 Generalized Cantor sets

Statement: Generalized Cantor sets

Suppose that we construct a new kind of Cantor set, by removing the middle half of each subinterval, rather than the middle third.

There is a subtlety concerning multiple eigenvalues: if the geometric multiplicity of an eigenvalue is lower than its algebraic multiplicity, then stability requires the eigenvalue to be strictly inside the unit disk.
a. Show that the length of the resulting set still vanishes, same as for the regular Cantor set.

b. Find the similarity dimension of the set.

c. Generalize the construction so as to produce a Cantor set with zero length and with a similarity dimension that can be picked as any arbitrary number in $0 < d < 1$.

6 Hill equation problem #01

Statement: Hill equation problem #01

Define the periodic (of period $2\pi$) square wave $S = S(\xi)$ as follows

$$S(\xi + 2n\pi) = 1 \text{ for } 0 < \xi < \pi \quad \text{and} \quad S(\xi + 2n\pi) = 0 \text{ for } \pi < \xi < 2\pi,$$  \hspace{1cm} (6.1)

where $n$ is an arbitrary integer. Consider now the Hill equation problem

$$\ddot{x} + a^2 S(\omega t) x = 0,$$  \hspace{1cm} (6.2)

where $a > 0$ and $\omega > 0$ are constants — note that the period is $T = \frac{2\pi}{\omega}$. Problem tasks:

1. Write the equations in the standard form $\dot{X} = A(\omega t) X$, where $A$ is a $2 \times 2$ matrix with period $2\pi$ and $X$ is a two-vector.

2. Find the fundamental solution $U$ to the system derived in item 1 — the matrix solution with $U(0) = \text{identity}$. 

3. Compute the Floquet matrix $R = U(T)$. An exact expression is possible.

4. Write the Floquet multipliers in terms of $\alpha = \frac{1}{2} \text{Tr}(R)$. You should find that $\alpha$ is a function of a single combination of parameters: $\tau > 0$. That is: $\alpha = \alpha(\tau)$. What is $\tau$?

5. Plot $\alpha$ versus $\tau$.

6. As $\tau$ grows from zero, there is a first threshold for instability: $\tau = \tau_1$ — this means that the solutions to (6.2) are bounded for $0 < \tau < \tau_1$, but some solutions grow exponentially for $\tau$ slightly above $\tau_1$. Compute $\tau_1$ (at least two significant digits).

7. For $\tau$ slightly above $\tau_1$, there is a solution to (6.2) that grows (slowly if $\tau$ is close to $\tau_1$). What is the frequency and period of this solution when $0 < \tau - \tau_1 \ll 1$?

**Hint:** The Floquet solutions have the general form $x = \chi(\omega t) e^{\mu t}$, where $\chi$ is periodic of period $2\pi$, and $\mu$ is the Floquet exponent. If $\mu$ is pure imaginary, and rationally related to $\omega$, this solution is periodic — with some period $T_s$ and frequency $2\pi/T_s$. If $\mu$ has a small positive real part, and its imaginary part limits to a rational fraction of $\omega$, then the situation described in item 7 arises.

---

4 In case you wonder: There is no problem with the discontinuity in $S$ as far as existence and uniqueness because $a^2 S x$ is Lipschitz continuous with respect to $x$ — which is what matters.
7 Simple Poincaré Map for a limit cycle #01

Statement: Simple Poincaré Map for a limit cycle #01

Consider the following autonomous phase plane system

\[
\begin{align*}
\frac{dx}{dt} &= (x^2 + y^4) \left( x - \frac{1}{4} x^3 - x^2 y - x y^2 - 4 y^3 \right), \\
\frac{dy}{dt} &= (x^2 + y^4) \left( y + \frac{1}{4} x^3 - \frac{1}{4} x^2 y + x y^2 - y^3 \right).
\end{align*}
\] (7.1)

This system has a periodic solution (show this), which can be written in the form

\[
x = 2 \cos \Phi, \quad y = \sin \Phi, \quad \text{where} \quad \frac{d\Phi}{dt} = 2 \left( x^2 + y^4 \right) = 2 \left( 1 + \cos^2 \Phi \right)^2.
\] (7.2)

This solution produces an orbit going through the point \(x = 0, y = 1\) in the phase plane. The orbit is an ellipse, as (7.2) shows.\(^5\)

Construct (either analytically or numerically) a Poincaré map near this orbit, and use it to show that the orbit is a stable limit cycle. Define the Poincaré map \(z \rightarrow u = P(z)\) as follows:

- For every sufficiently small \(z\), let \(x = X(t, z)\) and \(y = Y(t, z)\) be the solution of (7.1) defined by \(X(0, z) = 0\) and \(Y(0, z) = 1 + z\).

- For this solution the polar angle \(\theta\) in the phase plane is an increasing function of time, starting at \(\theta = \frac{1}{2} \pi\) for \(t = 0\). Thus, there is a time \(t = t_z\) at which the solution reaches \(\theta = \frac{5}{2} \pi\) (note that \(t_z\) is a function of \(z\)). Then take \(u = Y(t_z, z) - 1\).

Hint. Because \(t_z\) is a function of \(z\), unknown a priori, the definition of the Poincaré map above is a bit awkward to implement. To avoid having to calculate \(t_z\) for each solution, it is a good idea to use a parameter other than time to describe the orbits. For example, if the equations are written in terms of a parameter such as the polar angle — namely \(\frac{d\Phi}{dt} = F(x, y)\) and \(\frac{dy}{dt} = G(x, y)\), then the Poincaré map is easier to describe, as \(\theta\) varies from \(\theta = \frac{1}{2} \pi\) to \(\theta = \frac{5}{2} \pi\) in every one of the orbits needed to compute \(u = P(z)\). Note that this is just a “for example”, using the polar angle is not the best choice. Scale the variables first, so that the limit circle is a circle, not an ellipse.

Small challenge: You should be able to write \(P\) analytically. The formula is not even messy.

8 Problem 14.10.03 - Newton’s method in the complex plane

Statement for problem 14.10.03

Suppose that you want to solve an equation, \(g(x) = 0\). Then you can use Newton’s method, which is as follows: Assume that you have a “reasonable” guess, \(x_0\), for the value of a root. Then the sequence \(x_{n+1} = f(x_n)\), \(n \geq 0\), where \(f(x) = x - \frac{g(x)}{g'(x)}\), (8.1) converges (very fast) to the root.

Remark 8.2 (The idea). Assume an approximate solution \(g(x_a) \approx 0\). Write \(x_0 = x_a + \delta x\) to improve it, where \(\delta x\) is small. Then \(0 = g(x_0 + \delta x) \approx g(x_a) + g'(x_a) \delta x \Rightarrow \delta x \approx -\frac{g(x_a)}{g'(x_a)}\), and (8.1) follows.

Of course, if \(x_0\) is not close to a root, the method may not converge. Even if it converges, it may converge to a root that is far away from \(x_0\), not necessarily the closest root. In this problem we investigate the behavior of Newton’s method in the complex plane, for arbitrary starting points.

\(^5\) Note that \(\Phi\) is a strictly increasing function of time.
Consider iterations of the map in the complex plane generated by Newton’s method for the roots of $z^3 - 1 = 0$. That is

$$z_{n+1} = f(z_n) = \left(\frac{2}{3} + \frac{1}{3z_n^2}\right)z_n, \quad n \geq 0,$$

where $0 < |z_0| < \infty$ is arbitrary. Note that

$$\zeta_1 = 1, \quad \zeta_2 = e^{i2\pi/3} = \frac{1}{2}(-1 + i\sqrt{3}), \quad \text{and} \quad \zeta_3 = e^{i4\pi/3} = \frac{1}{2}(-1 - i\sqrt{3}),$$

are the roots of $z^3 = 1$.

**Your tasks:** Write a computer program to calculate the orbits $\{z_n\}_{n=0}^{\infty}$. Then, for every initial point $z_0$, draw a colored dot at the position of $z_0$, where the colors are picked as follows:

- $z_n \rightarrow \zeta_1$, cyan.
- $z_n \rightarrow \zeta_2$, magenta.
- $z_n \rightarrow \zeta_3$, yellow.
- No convergence, black.

What do you see? Do blow ups of the limit regions between zones.

**Hint.** Deciding that the sequence converges is easy: once $z_n$ gets “close enough” to one of the roots, then the very design of Newton’s method guarantees convergence. Thus, given a $z_0$, compute $z_N$ for some large $N$, and check if $|z_N - \zeta_j| < \delta$ for one of the roots and some “small” tolerance $\delta$ — which does not have to be very small, in fact $\delta = 0.25$ is good enough. You can get pretty good pictures with $N = 50$ iterations on a $150 \times 150$ grid. A larger $N$ is needed when refining near the boundary between zones.

**Hint.** If you use MatLab, do not plot “points”. Instead, plot “regions”, where the color of each pixel is decided by $z_0$ — use the command `image(x, y, C)` to plot. Why? Because using points leaves a lot of unpainted space in the figure, and gives much larger file sizes.

9 Problem 11.01.01 - Strogatz. Cantor’s not-countable argument

**Statement for problem 11.01.01**

Why is it that the argument used in Example 11.1.4 cannot be used to show that the rationals are also uncountable? After all, all rationals can be represented as decimals too.

10 Problem 11.03.08 - Strogatz. Sierpinski’s carpet

**Statement for problem 11.03.08**

Consider the process shown in figure 10.1. The closed unit box is divided into nine equal boxes, and the open central box is deleted. Then this process is repeated for each of the eight remaining sub-boxes, and so on. Figure 10.1 shows the first two stages.

A. Sketch the next stage, $S_3$.

B. Find the similarity dimension of the limiting fractal, known as the Sierpinski carpet.

C. Show that the Sierpinski carpet has zero area.

---

6 Numerically this means: choose a sufficiently fine grid in a rectangle, and pick every point in the grid. For example, select the square $-2 < x < 2$ and $-2 < y < 2$, where $z_0 = x + iy$. 

Figure 10.1: Problem 11.3.8. Sierpinski carpet construction. The areas shaded in black are the parts of the original square deleted at each stage of the fractal’s construction.

THE END.