A linear center that is a nonlinear spiral.

Consider the system
\[
\begin{align*}
\dot{x} &= -y - x^3 \\
\dot{y} &= x,
\end{align*}
\] (1.1)

For this system linearization predicts a center at the origin. However, by transforming the system into polar coordinates, it is easy to see that the origin is actually a nonlinear spiral. **Task #1: do this.**

**Task #2.** Implement a two-time expansion for solutions near the critical point, and calculate the rate of approach of the solutions towards the critical point as \( t \to \infty \). Is it exponential or algebraic? What is the rate (if exponential) exponent (if algebraic)?

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Note: The next problem has some challenging parts.

A single critical point between two limit cycles #1

Consider a phase plane system
\[
\dot{x} = f(x, y) \quad \text{and} \quad \dot{y} = g(x, y),
\] (2.1)

where \( f \) and \( g \) are smooth functions of all of its arguments. Assume that:

1. The system has isolated critical points only.

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1 The small parameter here is the radius \( \epsilon \) of the neighborhood considered.
2. The system has exactly two limit cycles, $\Gamma_1$ and $\Gamma_2$, with $\Gamma_2$ enclosing $\Gamma_1$.

3. Between $\Gamma_1$ and $\Gamma_2$ there is a single critical point $C$.

Given the above:

A. Calculate the index $I$ for $C$.

B. If $A$ is the matrix of the linearized system near $C$, show that $\Delta = \det(A) = 0$.

C. Construct an example of this situation, and sketch its phase plane portrait. Hint h2 below leads you to the construction of a system of this form that will have a cycle graph (produced by an homoclinic orbit connecting $C$ to itself). Sketch a phase plane portrait of the neighborhood of $C$. How would you “classify” this critical point?

Hints:

h1. The index of a critical point is 1 if $\Delta > 0$. This follows because then the critical point is either a node, a spiral point, satisfies $\Delta = \tau^2/4$ (linearly degenerate node), or has $\tau = 0$ (linearized center). In the first two cases the answer is obvious. For the other two cases an arbitrarily small perturbation can be used to transform the critical point into a spiral point, and then the continuity of the index yields the desired result.

h2. Consider systems of the form $\dot{x} = ax - by$ and $\dot{y} = bx + ay$, which are equivalent to $\dot{r} = ra$ and $\dot{\theta} = b$, where $r$ and $\theta$ are the polar coordinates (show this!) To do part C select $a = G(x^2 + y^2)$ and $b = b(x, y)$ appropriately. For example: take $b$ such that $\dot{\theta} = b > 0$ everywhere, except for a single point $C$ between the circles $r = 1$ and $r = 3$. Then choose $a$ so that limit cycles occur at $r = 1$ and $r = 3$, and so that $C$ is a critical point. The reason for selecting $a = G(r^2)$ as a function of $r^2$, and not $r$, is to avoid possible singular behavior at $r = 0$, where $r = r(x, y) = \sqrt{x^2 + y^2}$ is not differentiable.

3 Dulac’s criterion on a two holes region

Statement: Dulac’s criterion on a two holes region

Assume that the hypothesis of Dulac’s criterion hold for phase plane system in some region $R$. However, suppose that the region $R$ is not simply connected but has, exactly, two holes in it — e.g., the unit disk from which we exclude (i) a disk of radius 1/4, centered at $(x, y) = (0.5, 0)$, and (ii) a disk of radius 1/4, centered at $(x, y) = (-0.5, 0)$. We now “label” the holes: hole #1 and hole #2. Using Green’s theorem, show that

a. There is, at most, one closed orbit$^3$ in $R$ that encloses hole #1, but not hole #2.

b. There is, at most, one closed orbit$^3$ in $R$ that encloses hole #2, but not hole #1.

c. There is, at most, one closed orbit$^3$ in $R$ that encloses both holes.

d. No other closed orbits are possible.

It follows that there can be only three “types” of closed orbits within $R$; but how many of them can co-exist?

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$^2$ Here $\tau$ is the trace of the linearization matrix at the critical point, while $\Delta$ is the determinant.

$^3$ A limit cycle.
4 Liapunov Function # 01

Statement: Liapunov Function # 01

Show that the system
\[
\frac{dx}{dt} = -x + 2y^3 - 2y^4 \quad \text{and} \quad \frac{dy}{dt} = -x - y + xy,
\] (4.1)
has no periodic solutions.

Hint. Find a Liapunov function. Try the form \( L = x^m + ay^n \).

5 Multiple scales and limit cycles #01

Statement: Multiple scales and limit cycles #01

Consider the equation
\[
\frac{d^2 x}{dt^2} - \epsilon \cos x \frac{dx}{dt} + \frac{1}{\sqrt{\epsilon}} \sin(\sqrt{\epsilon} x) = 0, \quad \text{where} \quad 0 < \epsilon \ll 1. \] (5.1)

Use a multiple scales analysis to calculate the frequency, stability and amplitude of any limit cycle (the frequency up to the first correction beyond linear and the amplitude up to leading order).

THE END.