Consider a system made by two coupled oscillators, where each of the oscillators (when not coupled) has a stable attracting limit cycle. Let the limit cycle solutions for the two oscillators be given by $\vec{x}_1 = \vec{F}_1(\omega_1 t)$ and $\vec{x}_2 = \vec{F}_2(\omega_2 t)$, where $\vec{x}_1$ and $\vec{x}_2$ are the vectors of variables for each of the two systems, the $\vec{F}_j$ are periodic functions of period $2\pi$, and the $\omega_j$ are constants (related to the limit cycle periods by $\omega_j = 2\pi/T_j$). In the un-coupled system, the two limit cycle orbits make up a stable attracting invariant torus for the evolution. Assume now that either the coupling is weak, or that the two limit cycles are strongly stable. Then the stable attracting invariant torus survives for the coupled system. The solutions (on this torus) can be (approximately) represented by

$$\vec{x}_1 \approx \vec{F}_1(\theta_1) \quad \text{and} \quad \vec{x}_2 \approx \vec{F}_2(\theta_2),$$

with a (slightly) changed shape and position.
where $\theta_1 = \theta_1(t)$ and $\theta_2 = \theta_2(t)$ satisfy some equations, of the general form

$$\dot{\theta}_1 = \omega_1 + K_1(\theta_1, \theta_2) \quad \text{and} \quad \dot{\theta}_2 = \omega_2 + K_2(\theta_1, \theta_2).$$

(1.2)

Here $K_1$ and $K_2$ are the “projections” of the coupling terms along the oscillator limit cycles. For example, take $K_1(\theta_1, \theta_2) = \sin \theta_1 \cos \theta_2$ and $K_2(\theta_1, \theta_2) = \sin \theta_2 \cos \theta_1$. Another example is the one in § 8.6 of Strogatz’ book (Nonlinear Dynamics and Chaos), where a model system with

$$K_1(\theta_1, \theta_2) = -\kappa_1 \sin(\theta_1 - \theta_2) \text{ and } K_2(\theta_1, \theta_2) = \kappa_2 \sin(\theta_1 - \theta_2)$$

is introduced, with constants $\kappa_1, \kappa_2 > 0$. Note that:

1. In (1.2), $K_1$ and $K_2$ must be $2\pi$-periodic functions of $\theta_1$ and $\theta_2$.

2. The phase space for (1.2) is the invariant torus $T$, on which $\theta_1$ and $\theta_2$ are the angles. We can also think of $T$ as a $2\pi \times 2\pi$ square with its opposite sides identified. On $T$ a solution is periodic if and only if $\theta_1(t + T) = \theta_1(t) + 2n\pi$ and $\theta_2(t + T) = \theta_2(t) + 2m\pi$, where $T > 0$ is the period, and both $n$ and $m$ are integers.

3. In the “Coupled oscillators # 01” problem an example of the process leading to (1.2) is presented.

4. The $\theta_j$’s are the oscillator phases. One can also define oscillator frequencies, even when the $\theta_j$’s do not have the form $\theta_j = \omega_j t$, with $\omega_j$ constant.

   The idea is that, near any time $t_0$ we can write

   $$\theta_j = \theta_j(t_0) + \dot{\theta}_j(t_0) (t - t_0) + \ldots,$$

   identifying $\dot{\theta}_j(t_0)$ as the local frequency. Hence, we define the oscillator frequencies by $\tilde{\omega}_j = \dot{\theta}_j$. These frequencies are, of course, generally not constants.

5. The notion of phases can survive even if the limit cycles cease to exist (i.e.: oscillator death). For example: if the equations for $\theta_1$ and $\theta_2$ have an attracting critical point. We will see examples where this happens in the problems, e.g.: “Bifurcations in the torus # 01”.

1.2 Phase locking and oscillator death

The coupling of two oscillators, each with a stable attracting limit cycle, can produce many behaviors. Two of particular interest are

1. Often, if the frequencies are close enough, the system phase locks. This means that a stable periodic solution arises, in which both oscillators run at some composite frequency, with their phase difference kept constant. The composite frequency need not be constant. In fact, it may periodically oscillate about a constant average value.

2. However, the coupling may also suppress the oscillations, with the resulting system having a stable steady state. This even if none of the component oscillators has a stable steady state. This is oscillator death. It can happen not only for coupled pairs of oscillators, but also for chains of oscillators with coupling to the nearest neighbors.

On the other hand, we note that it is also possible to produce an oscillating system, with a stable oscillation, by coupling non-oscillating systems (e.g., the coupling of excitable systems can do this).

2 Notes: first order equation with a periodic right hand side

You will be asked to justify the statements below in one of the problems. They are stated here because these results are used/needed in the answers to some of the other problems.
Consider the equation
\[ \dot{\phi} = 1 - \kappa \sin \phi, \quad \text{where } 0 < \kappa < 1. \] (2.1)

Since \( \dot{\phi} \geq 1 - \kappa > 0 \), \( \phi \) is monotone increasing. It can be shown that:

1. There is a constant \( 0 < \mu < 1 \), and a function \( \Phi = \Phi(\zeta) \) — periodic of period \( 2\pi \) — such that any solution to (2.1) has the form
\[ \phi = \mu (t - t_0) + \Phi(\mu (t - t_0)), \] (2.2)
where \( t_0 \) is a constant and \( \Phi(0) = 0 \). It follows that \( \sin(\phi) \) is periodic in \( t \), with period \( T = \frac{2\pi}{\mu} \).

2. The period-average \( M \) for \( \sin(\phi) \) is given by
\[ M = \text{average}(\sin \phi) = \frac{1-\mu}{\kappa} > 0. \] (2.3)

3. Let \( \phi_* \) be the solution to (2.1) defined by \( \phi_*(0) = 0 \) — i.e.: set \( t_0 = 0 \) in (2.2). Then
\[ \Theta(\mu t) = \int_0^t (\sin(\phi_*(s)) - M) \, ds = -\frac{1}{\kappa} \Phi(\mu t), \] (2.4)
where \( \Theta \) is defined by the first equality.

4. Assume that \( 0 < \kappa \ll 1 \). Then a Poincaré-Lindstedt expansion yields
\[ \phi_* = \mu t - \kappa (1 - \cos(\mu t)) + O(\kappa^2) \quad \text{and} \quad \mu = 1 - \frac{1}{2} \kappa^2 + O(\kappa^4). \] (2.5)

It follows that \( T = 2\pi + \pi \kappa^2 + O(\kappa^4) \) and \( M = \frac{1}{2} \kappa + O(\kappa^3) \).

5. Assume that \( 0 < 1 - \kappa \ll 1 \). Then
\[ \mu = O(\sqrt{1 - \kappa}) \] (2.6)

### 3 Attracting Lines in the Phase Plane #01

**Statement: Attracting Lines in the Phase Plane #01**

You may have observed the frequent appearance of special curves along which solutions tend to bunch up. Sometimes these are associated with the stable and/or unstable manifolds of certain critical points, and sometimes they are not. Below is a particular example, where you are expected to justify the special line with a scaling/asymptotic kind of analysis (coupled with an appropriate qualitative argument on the phase plane). The problem is aimed at testing that you can use these sort of tools properly, so do it using them. If you think of another way of doing it, and you want to show off, then I will look at it and consider it for extra credit — but this can backfire if this “other way” contains some grave conceptual error (thus do it only if you are certain).

Consider the damped pendulum equation:
\[ \frac{d^2\theta}{dt^2} + a \frac{d\theta}{dt} + \sin \theta = 0, \] (3.1)
where \( a \gg 1. \) Show that ALL the trajectories in the phase plane \((\theta, \dot{\theta})\) end up, after a brief transient period, following the curve:
\[ \frac{d\theta}{dt} \approx -\frac{1}{a} \sin \theta. \] (3.2)

**Remark 3.1 (Warnings)**

\(^2\) Say, in the phase plane plots with the problem set answers, or if you played with the MatLab scripts in the course toolkit.
a. Do not fall into the trap of just showing that $\dot{\theta} + \frac{1}{a} \sin \theta$ is small! Both terms here are small, so showing this provides no new information. What you must show is that this expression is much smaller than either term, for example: $\dot{\theta} + \frac{1}{a} \sin \theta = O(a^{-2})$. In fact, it is possible to show that $\dot{\theta} + \frac{1}{a} \sin \theta = O(a^{-3})$.

b. Do not fall into the trap of arguing that, just because some derivative is multiplied by a small parameter, you can neglect it — this is not always true. The qualitative argument in the phase plane is the step that will allow you to justify something like this.

c. Show that all the trajectories (eventually) satisfy (3.2), not just that there are some that do.

4 Coupled oscillators #01 (derive phase equations)

Statement: Coupled oscillators #01

In this problem we present an example of the process described in § 1.1, and consider the coupling of two oscillators with a stable, and strongly attracting, limit cycle each. The oscillators are very simple, with trivial equations in polar coordinates. This simplifies the analysis enormously, but the principles illustrated here are valid for the coupling of more generic oscillators.

Consider the following equations for two coupled oscillators

$$
\begin{align*}
\dot{x}_j &= -\omega_j y_j + \lambda_j (R_j^2 - x_j^2 - y_j^2) x_j + F_j(x_1, y_1, x_2, y_2), \\
\dot{y}_j &= \omega_j x_j + \lambda_j (R_j^2 - x_j^2 - y_j^2) y_j + G_j(x_1, y_1, x_2, y_2),
\end{align*}
$$

where $j = 1$ or $j = 2$, and

(a) $\omega_j > 0$, $R_j > 0$ and $\lambda_j > 0$, are constants, with $\lambda_j \gg 1$,

(b) $F_j$ and $G_j$ are some functions — these are the coupling terms.

Using the fact that $\lambda_j \gg 1$, write reduced equations for the two phases $\theta_1$ and $\theta_2$, defined by $x_j = r_j \cos \theta_j$ and $y_j = r_j \sin \theta_j$, where $r_j = \sqrt{x_j^2 + y_j^2}$. In particular, consider the following cases

1. What form do the reduced equations take when $y_j F_j$ and $x_j G_j$ are only functions of the variables $\eta = x_1 x_2 + y_1 y_2$ and $\xi = y_1 x_2 - x_1 y_2$.

2. What form do the reduced equations take when $G_1 = G_2 = 0$, $F_1 = -\alpha \frac{R_1}{R_2} x_2$, and $F_2 = -\beta \frac{R_2}{R_1} x_1$ — where $\alpha$ and $\beta$ are constants.

Hint. Write the equations in polar coordinates. Then consider what happens in a neighborhood of the limit cycles for the two oscillators when de-coupled — i.e.: $r_j$ not too far from $R_j$. In this context, argue that the dependence on the radial variables can be made trivial.

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4 Examples where this fails are wide-spread in applications. They are not a mathematical curiosity.

4 Recall that $r_j \dot{r}_j = x_j \dot{x}_j + y_j \dot{y}_j$ and $r_j^2 \dot{\theta}_j = x_j \dot{y}_j - y_j \dot{x}_j$.

5 Use arguments similar to the one introduced to describe relaxation oscillations, e.g.: for the van der Pol equation. Another example occurs when justifying that inertial terms can be neglected in the limit of a large viscosity.
5 Friends on a circular track run in opposite directions

Statement: Friends on a circular track run in opposite directions

In § 8.6 of his book (Nonlinear Dynamics and Chaos) Strogatz introduces the following model equations in the torus

\[
\dot{\theta}_1 = \omega_1 - \kappa_1 \sin(\theta_1 - \theta_2) \quad \text{and} \quad \dot{\theta}_2 = \omega_2 + \kappa_2 \sin(\theta_1 - \theta_2),
\]

(5.1)

where: (a) \( \theta_1, \theta_2 \) are the phases of two oscillators, (b) \( \omega_1, \omega_2 > 0 \) are their natural frequencies, and (c) \( \kappa_1, \kappa_2 > 0 \) are the coupling constants. Then he gives a simple interpretation for the equations as modeling two friends jogging in a circular track of radius \( R \), where \((\theta_1, \theta_2)\) are their (angular) positions on the track, \((R\omega_1, R\omega_2)\) are their preferred running speeds, and the coupling models their desire to run together. In this interpretation (5.1) corresponds to the two friends running along the track in the same direction (counter-clockwise). Imagine now a situation where they are running in opposite directions along the track.7 Then the equations have to be modified to

\[
\dot{\theta}_1 = \omega_1 - \kappa_1 \sin(\theta_1 - \theta_2) \quad \text{and} \quad \dot{\theta}_2 = -\omega_2 + \kappa_2 \sin(\theta_1 - \theta_2).
\]

(5.2)

Equivalently, with \( \varphi_1 = \theta_1 \) and \( \varphi_2 = -\theta_2 \),

\[
\dot{\varphi}_1 = \omega_1 - \kappa_1 \sin(\varphi_1 + \varphi_2) \quad \text{and} \quad \dot{\varphi}_2 = -\omega_2 - \kappa_2 \sin(\varphi_1 + \varphi_2).
\]

(5.3)

The motivation given above for (5.2) is not-too-serious, but these equations — in the form given in (5.3) — are a particular example of the type of equations that result when very stable limit cycle oscillators are coupled — e.g., see the problem Coupled oscillators # 01.

Study the behavior of the system in (5.2). In particular: Find all the bifurcations. Is there phase locking? Note: since (5.2) is a system in the torus, bifurcations that do not appear in the plane might occur, look carefully!

Notation and units: To simplify the answer, assume that the time unit is such that \( \omega_1 + \omega_2 = 1 \). In addition, use the notation \( \kappa = \kappa_1 + \kappa_2 \) — note \( \kappa > 0 \).

Hint. Use the results in § 2, “Notes: first order equation with a periodic right hand side”. Do not try to pin-point the various regimes and bifurcations using explicit formulas involving \( \omega_1, \omega_2, \kappa_1 \) and \( \kappa_2 \) only — you will also need either \( M = M(\kappa) \) or \( \mu = \mu(\kappa) \), defined in § 2.

6 These equations appear in other contexts, as mentioned by Strogatz. Also see § 1.

7 Coach’s orders, because otherwise they run together, talk, and do not train properly.

6 Index for a linearly degenerate node

Statement: Index for a linearly degenerate node

Consider a phase plane system

\[
\dot{x} = f(x, y) \quad \text{and} \quad \dot{y} = g(x, y),
\]

(6.1)

where \( f \) and \( g \) are smooth functions of all of its arguments. Assume that:

1. The origin is a critical point. That is \( f(0, 0) = g(0, 0) = 0 \).

2. The origin is a linearly degenerate node. That is: if \( A \) is the \( 2 \times 2 \) matrix corresponding to the linearized system near the critical point at the origin, then \( A \) has two equal, real and nonzero, eigenvalues. Equivalently, if \( \tau = \text{tr}(A) \) and \( \Delta = \det(A) \), then \( 0 < \Delta = \tau^2/4 \).
Because $\Delta \neq 0$, the origin is an isolated critical point (inverse function theorem), and has an index associated with it. Let this index be $I_0$. **Your task here is to calculate** $I_0$.

**Note.** Degenerate nodes are structurally unstable. Hence you cannot calculate their index by simply calculating the index for the linearized system, you have to do a slightly more sophisticated calculation. Interestingly, even though the actual phase plane portrait for a degenerate node cannot be ascertained from the linearized system alone (structural instability), you need no nonlinear information to calculate the index!

**Hint:**

3. Write the system (6.1) in vector form

$$\dot{Y} = F(Y), \quad \text{where } Y = (x, y)^T \text{ and } F = (f, g)^T.$$

(6.2)

where $Y = (x, y)^T$ and $F = (f, g)^T$.

4. Assume that the matrix $A$ has the form

$$A = \begin{pmatrix} \mu & a \\ 0 & \mu \end{pmatrix},$$

(6.3)

where $\mu \neq 0$ is the double eigenvalue, and $a$ is some constant. There is no loss of generality here: any matrix with a double eigenvalue can be reduced to this form by an appropriate choice of coordinates.

5. Consider now the one parameter family of systems

$$\dot{Y} = F(Y) + \epsilon \sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Y, \quad 0 \leq \epsilon < \sqrt{\frac{a^2}{4} + \mu^2 - \frac{|a|}{2}},$$

(6.4)

where $\sigma = \text{sign}(a)$ if $a \neq 0$, and $\sigma = 1$ if $a = 0$. For these systems the origin is a critical point, with the linearization matrix

$$B = \begin{pmatrix} \mu & a + \epsilon \sigma \\ \epsilon \sigma & \mu \end{pmatrix}.$$  

In particular $\det(B) = \mu^2 - \epsilon (|a| + \epsilon) > 0,$

(6.5)

so that the origin is an isolated critical point (inverse function theorem).

Let now $I = I(\epsilon)$ be the index of the critical point at the origin for (6.4). Calculate $I(\epsilon)$ for $\epsilon > 0$, and use the properties of the index to get $I(0) = I_0$.

---

7 Phase Plane Surgery #01

**Statement: Phase Plane Surgery #01**

Can a smooth vector field exist in the plane such that:

- The critical points are $P_1 = (-2, 0)$, $P_2 = (0, 0)$ and $P_3 = (2, 0)$.
- All the critical points are spirals.
- The circles with radii: $R_1 = 1$ centered at $P_1$, $R_2 = 4$ centered at $P_2$, and $R_3 = 1$ centered at $P_3$, are orbits.

Would your answer change if $P_2$ is a saddle?

In either case, if your answer is yes, sketch the way the orbits might look in an example satisfying the criteria above.
Challenge question: In either case, if your answer is yes, can you give an actual example (i.e.: write the vector field explicitly) that gives you a phase portrait with the same qualitative features (the closed orbits need not be circles for this).

8 Example of a reversible system that is not conservative

Statement: Example of a reversible system that is not conservative

Give an example of a reversible system that is not conservative.

Hint. Remember that a phase plane system

\[ \begin{align*}
\dot{x} &= f(x, y) \quad \text{and} \quad \dot{y} = g(x, y), \\
\end{align*} \tag{8.1} \]

is reversible if, for example, \( f \) is odd and \( g \) is even in \( y \) — that is: \( f(x, -y) = -f(x, y) \) and \( g(x, -y) = g(x, y) \). In this case the change \( t \to -t \) and \( y \to -y \) leaves the system invariant.

In addition, we know that conservative systems cannot have sinks or sources. Now, ask yourself: what systems have exactly the opposite property [almost every critical point is either a source or a sink]. Then produce a system of this kind, with \( f \) odd and \( g \) even.

9 Problem 14.09.22 (Attracting and Liapunov stable)

Statement for problem 14.09.22

Recall the definitions for the various types of stability that concern critical points:

Let \( x^* \) be a fixed point of the system \( \dot{x} = f(x) \). Then:

1. \( x^* \) is attracting if there is a \( \delta > 0 \) such that \( \lim_{t \to \infty} \|x(0) - x^*\| < \delta \). That is: any trajectory that starts within \( \delta \) of \( x^* \) eventually converges to \( x^* \). Note that trajectories that start nearby \( x^* \) need not stay close in the short run, but must approach \( x^* \) in the long run.

2. \( x^* \) is Liapunov stable if for each \( \epsilon > 0 \), there is a \( \delta > 0 \) such that \( \|x(t) - x^*\| < \epsilon \) for \( t > 0 \), whenever \( \|x(0) - x^*\| < \delta \). Thus, trajectories that start within \( \delta \) of \( x^* \) stay within \( \epsilon \) of \( x^* \) for all \( t > 0 \).

In contrast with attracting, Liapunov stability requires nearby trajectories to remain close for all \( t > 0 \).

3. \( x^* \) is asymptotically stable if it is both attracting and Liapunov stable.

4. \( x^* \) is repeller if there are \( \epsilon > 0 \) and \( \delta > 0 \) such that if \( 0 < \|x(0) - x^*\| < \delta \), after some critical time \( \|x(t) - x^*\| > \epsilon \) applies (i.e., for \( t > t_c \)). Repellers are a special kind of unstable critical points.

For each of the following systems, decide whether the origin is attracting but not Liapunov stable, Liapunov stable, asymptotically stable, repeller, or unstable but not a repeller.

a) \( \dot{x} = 2y \) and \( \dot{y} = -3x \).

b) \( \dot{x} = y \cos(x^2 + y^2) \) and \( \dot{y} = -x \cos(x^2 + y^2) \).

c) \( \dot{x} = -x \) and \( \dot{y} = -|y| y \).

d) \( \dot{x} = 2xy \) and \( \dot{y} = y^2 - x^2 \).

e) \( \dot{x} = x - 2y x^2 - 4y^3 \) and \( \dot{y} = y + x^3 + 2xy^2 \). \hspace{1cm} \text{Hint: what happens along } x = 0? \)
f) \( \dot{x} = y \) and \( \dot{y} = x \).
g) Finally, consider the critical point \((x, y) = (1, 0)\), for the system
\[
\dot{x} = (1 - r^2) x - (1 - \frac{x}{r}) y \quad \text{and} \quad \dot{y} = (1 - r^2) y + (1 - \frac{y}{r}) x,
\] (9.1)

defined in the “punctured” plane \(r = \sqrt{x^2 + y^2} > 0\). Hint: write the equations in polar coordinates.

Additional hints. In some cases you can get the answer by finding a function \(J = J(x, y)\) with a local minimum at the origin such that \(\frac{dJ}{dt} > 0\) along trajectories — or maybe one such \(\frac{dJ}{dt} < 0\), or maybe one such \(\frac{dJ}{dt} = 0\). In other cases look for special trajectories that either leave, or approach, the origin.

10 Problem 06.01.08 - Strogatz (Computer generated phase portrait)

Statement for problem 06.01.08

Plot a computer generated phase plane portrait for the van der Pol oscillator
\[
\frac{dx}{dt} = y, \quad \text{and} \quad \frac{dy}{dt} = -x + y(1 - x^2).
\] (10.1)

11 Two limit cycles and enclosed critical points

Statement: Two limit cycles and enclosed critical points

Consider a phase plane system of the form
\[
\frac{dx}{dt} = f(x, y) \quad \text{and} \quad \frac{dy}{dt} = g(x, y),
\] (11.1)

where \(f\) and \(g\) are both smooth. Assume that:

A. The circles \(r = 1\) and \(r = 2\) are limit cycles.
B. There is a saddle point at \((x, y) = (1.5, 0)\).

Show that there must be at least one more critical point in the annular region \(1 < r < 2\) between the two limit cycles.