Due last day of classes.
Turn it in (by 3PM) at the Math. Problem Set Boxes, right outside ................. room 4-174.
There is a box/slot there for 385. Be careful to use the right box (there are many slots). This
problem set is optional, and intended for those that missed a problem set because of an excused
reason.

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1 Bifurcations of a Critical Point for a 1-D map

Statement: Bifurcations of a Critical Point for a 1D map

When studying bifurcations of critical points in the lectures, we argued that we could understand most of the
relevant possibilities simply by looking at 1-D flows. The argument was based on the idea that, generally, a stable
critical point will become unstable in only one direction at a bifurcation — so that the flow will be trivial in all the
other directions, and we need to concentrate only on what occurs in the unstable direction. The only important
situation that is missed by this argument, is the case when two complex conjugate eigenvalues become unstable
(Hopf bifurcation). Because in real valued systems eigenvalues arise either in complex conjugate pairs or as single
real ones, the cases where only one (or a complex pair) go unstable are the the only ones likely to occur (barring
situations with special symmetries that “lock” eigenvalues into synchronous behavior).

The same argument can be made when studying bifurcations of limit cycles in any number of dimensions. In
this case one considers the Poincaré map near the limit cycle, with the role of the eigenvalues taken over by the
Floquet multipliers. Again, we can argue that we can understand a good deal of what happens by replacing the
(multi-dimensional) Poincaré map by a one dimensional map with a stable fixed point, and then asking what can
happen if the fixed point changes stability as some parameter is varied. In this problem you will be asked to do
this.

Remark 1.0.1 Some important cases are missed by this approach: the case where a pair of complex Floquet
multipliers becomes unstable (Hopf bifurcation of a limit cycle), and the cases where a bifurcation occurs because of
an interaction of the limit cycle with some other object (e.g., a critical point). Several examples of these situations
can be found in section 8.4 of Strogatz’ book (Global Bifurcations of Cycles).

Consider a one dimensional (smooth) map from the real line to itself

\[ x \rightarrow y = f(x, \mu) \]  

that depends on some (real valued) parameter \( \mu \). **Assume that \( x = 0 \) is a fixed point for all values of \( \mu \) — that is, \( f(0, \mu) \equiv 0 \).** Furthermore, assume that \( x = 0 \) is stable for \( \mu < 0 \) and unstable for \( \mu > 0 \). That is:

\[
\left| \frac{\partial f}{\partial x}(0, \mu) \right| < 1 \quad \text{for } \mu < 0, \quad \text{and} \\
\left| \frac{\partial f}{\partial x}(0, \mu) \right| > 1 \quad \text{for } \mu > 0.
\]
A further assumption, that involves no loss of generality (since the parameter \( \mu \) can always be re-defined to make it true) is that

\[
\frac{\partial^2 f}{\partial x \partial \mu}(0, 0) \neq 0. \tag{1.4}
\]

This guarantees that the loss of stability is linear in \( \mu \), as \( \mu \) crosses zero. This is what is called a \textit{transversality condition}. It means this:

Graph of the Floquet multiplier \( \frac{\partial f}{\partial x}(0, \mu) \) as a function of \( \mu \). Then the resulting curve crosses one of the lines \( y = \pm 1 \) transversally (curves not tangent at the common point) for \( \mu = 0 \).

By doing an appropriate expansion of the map \( f \) for \( x \) and \( \mu \) small (or by any other means), show that (generally\(^2\)) the following happens:

\[\text{a. For } \frac{\partial f}{\partial x}(0, 0) = 1, \text{ either:}\]

\[\text{a1. Transcritical bifurcation (no special symmetries assumed for } f)\text{: There exists another fixed point, } x_* = x_*(\mu) = O(\mu), \text{ such that: } x_* \neq 0 \text{ is unstable for } \mu < 0 \text{ and } x_* \neq 0 \text{ is stable for } \mu > 0. \text{ The two points “collide” at } \mu = 0 \text{ and exchange stability.}\]

\[\text{a2. Supercritical or soft pitchfork bifurcation, assuming that } f \text{ is an odd function of } x\text{: Two stable fixed points exist for } \mu > 0, \text{ one on each side of } x = 0, \text{ at a distance } O(\sqrt{\mu}). \text{ All three points merge for } \mu = 0.\]

\[\text{a3. Subcritical or hard pitchfork bifurcation, assuming that } f \text{ is an odd function of } x\text{: Two unstable fixed points exist for } \mu < 0, \text{ one on each side of } x = 0, \text{ at a distance } O(\sqrt{-\mu}). \text{ All three points merge for } \mu = 0.\]

What does all this mean in the context of the Poincaré map for a limit cycle?

\[\text{b. For } \frac{\partial f}{\partial x}(0, 0) = -1 \text{ (no special symmetries assumed for } f), \text{ either:}\]

\[\text{b1. Supercritical or soft flip bifurcation: For } \mu > 0 \text{ two points } x_1(\mu) \approx -x_2(\mu) = O(\sqrt{\mu}) \text{ exist, on each side of the fixed point } x = 0, \text{ with } x_2 = f(x_1, \mu) \text{ and } x_1 = f(x_2, \mu). \text{ Thus } \{x_1, x_2\} \text{ is a period two orbit for the map (1.1). Show that this orbit is stable.}\]

\[\text{b2. Subcritical or hard flip bifurcation: For } \mu < 0 \text{ two points } x_1(\mu) \approx -x_2(\mu) = O(\sqrt{-\mu}) \text{ exist, on each side of the fixed point } x = 0, \text{ with } x_2 = f(x_1, \mu) \text{ and } x_1 = f(x_2, \mu). \text{ Thus } \{x_1, x_2\} \text{ is a period two orbit for the map (1.1). Show that this orbit is unstable.}\]

In the context of the Poincaré map for a limit cycle, a flip bifurcation corresponds to a period doubling bifurcation of the limit cycle.

\[\text{Hint 1.0.1 If you expand } f \text{ in a Taylor expansion near } x = 0 \text{ and } \mu = 0, \text{ up to the leading order beyond the trivial first term } (f \sim \pm x), \text{ and you make sure to keep all the relevant terms (and nothing else), and you make sure to identify certain terms that must vanish, the problem reduces to (mostly) trivial high-school algebra. In figuring out what to keep and what not to keep, note that you have two small quantities } (x \text{ and } \mu), \text{ whose sizes are related. The process is very similar to the Hopf bifurcation expansion calculation (e.g. see course notes), but much simpler computationally.}\]

For part (b) you will have to keep in the expansion not just the leading order terms beyond the trivial first term, but the next order as well. \textit{Reason: In this case, you will be looking for solutions to the equation } f(f(x, \mu), \mu) = x. \textit{ But, when you calculate } f(f(x, \mu), \mu), \textit{ you will see that the second order terms cancel out — thus the need for an extra term in the expansion. This is the same phenomena that forces the Hopf bifurcation calculation to third order.}\]

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\(^2\) There are special conditions under which all this fails. You must find them as part of your analysis. What are they?
VERY IMPORTANT.

To standardize the notation, use the following symbols in your answer:

\[ \nu = \frac{\partial f}{\partial x}(0, 0) \] — note that \( \nu = \pm 1 \),

\[ a = \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(0, 0), \quad b = \frac{\partial^2 f}{\partial x \partial \mu}(0, 0), \quad c = \frac{1}{6} \frac{\partial^3 f}{\partial x^3}(0, 0), \] and \[ d = \frac{1}{2} \frac{\partial^3 f}{\partial x^2 \partial \mu}(0, 0). \]

Then obtain leading order expressions for the fixed points and flip-bifurcation orbits in terms of these quantities.

2 Sierpinski gasket

Statement: Sierpinski gasket

Consider the fractal (a “Sierpinski gasket”) in the plane, made in the following recursive fashion:

1. Start with an equilateral triangle, with sides of length \( L \).
2. Draw the lines joining the sides mid-points, and divide it into four equal equilateral sub-triangles.
3. Remove the sub-triangle at the center.
4. Repeat the process with each of the other three remaining sub-triangles.

**Figure 2.1:** The picture on the right illustrates the recursion, showing the result of the first iteration in the process described above.

Now, do the following:

A. Calculate the box dimension of the fractal.
B. Calculate the self-similar dimension of the fractal.
C. Calculate the surface area of the fractal.
D. Show that the fractal has as many points as a full square — **This part is hard(er).**
E. Let \( d_s \) be the dimension calculated in part A. Modify the construction of the fractal, in such a way that the modified fractal can be selected to have any given box dimension \( 0 < d < d_s \).
   **Hint:** take out bigger chunks at each stage.
F. Construct fractals (subsets of the plane) such that their box dimensions can be selected to have any given box dimension \( d_s < d < 2 \).

THE END.