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1 Notes: coupled oscillators, phase locking, oscillator death, etc.

1.1 On phases and frequencies

Consider a system made by two coupled oscillators, where each of the oscillators (when not coupled) has a stable attracting limit cycle. Let the limit cycle solutions for the two oscillators be given by \( \vec{x}_1 = \vec{F}_1(\omega_1 t) \) and \( \vec{x}_2 = \vec{F}_2(\omega_2 t) \), where \( \vec{x}_1 \) and \( \vec{x}_2 \) are the vectors of variables for each of the two systems, the \( \vec{F}_j \) are periodic functions of period \( 2\pi \), and the \( \omega_j \) are constants (related to the limit cycle periods by \( \omega_j = 2\pi/T_j \)). In the uncoupled system, the two limit cycle orbits make up a \textit{stable attracting invariant torus} for the evolution. Assume now that either the coupling is weak, or that the two limit cycles are strongly stable. \textit{Then the stable attracting invariant torus survives for the coupled system}. The solutions (on this torus) can be (approximately) represented by

\[
\vec{x}_1 \approx \vec{F}_1(\theta_1) \quad \text{and} \quad \vec{x}_2 \approx \vec{F}_2(\theta_2),
\]

where \( \theta_1 = \theta_1(t) \) and \( \theta_2 = \theta_2(t) \) satisfy some equations, of the general form

\[
\dot{\theta}_1 = \omega_1 + K_1(\theta_1, \theta_2) \quad \text{and} \quad \dot{\theta}_2 = \omega_2 + K_2(\theta_1, \theta_2).
\]

(1.1)

Here \( K_1 \) and \( K_2 \) are the “projections” of the coupling terms along the oscillator limit cycles. For example, take \( K_1(\theta_1, \theta_2) = \sin \theta_1 \cos \theta_2 \) and \( K_2(\theta_1, \theta_2) = \sin \theta_2 \cos \theta_1 \). Another example is the one in § 8.6 of Strogatz’ book \textit{(Nonlinear Dynamics and Chaos)}, where a model system with

\[
K_1(\theta_1, \theta_2) = -\kappa_1 \sin(\theta_1 - \theta_2) \quad \text{and} \quad K_2(\theta_1, \theta_2) = \kappa_2 \sin(\theta_1 - \theta_2)
\]

is introduced, with constants \( \kappa_1, \kappa_2 > 0 \). Note that:

1. In (1.2), \( K_1 \) and \( K_2 \) must be \( 2\pi \)-periodic functions of \( \theta_1 \) and \( \theta_2 \).
2. The phase space for (1.2) is the invariant torus \( \mathcal{T} \), on which \( \theta_1 \) and \( \theta_2 \) are the angles. We can also think of \( \mathcal{T} \) as a \( 2\pi \times 2\pi \) square with its opposite sides identified. On \( \mathcal{T} \) a solution is periodic if and only if
\[
\theta_1(t + T) = \theta_1(t) + 2n\pi \quad \text{and} \quad \theta_2(t + T) = \theta_2(t) + 2m\pi,
\]

where \( T > 0 \) is the period, and both \( n \) and \( m \) are integers.
3. In the “Coupled oscillators # 01” problem an example of the process leading to (1.2) is presented.
4. The \( \theta_j \)'s are the oscillator phases. One can also define oscillator frequencies, even when the \( \theta_j \)'s do not have the form \( \theta_j = \omega_j t \), with \( \omega_j \) constant. The idea is that, near any time \( t_0 \) we can write \( \theta_j = \hat{\theta}_j(t_0) + \dot{\theta}_j(t_0)(t - t_0) + \ldots \), identifying \( \dot{\theta}_j(t_0) \) as the local frequency. Hence, we define the oscillator frequencies by \( \tilde{\omega}_j = \hat{\theta}_j \). These frequencies are, of course, \textit{generally not constants}.
5. The notion of phases can survive \textit{even if} the limit cycles cease to exist (i.e.: oscillator death). For example: if the equations for \( \theta_1 \) and \( \theta_2 \) have an attracting critical point. We will see examples where this happens in the problems, e.g.: “Bifurcations in the torus # 01”.

\footnote{With a (slightly) changed shape and position.}
1.2 Phase locking and oscillator death
The coupling of two oscillators, each with a stable attracting limit cycle, can produce many behaviors. Two of particular interest are

1. Often, if the frequencies are close enough, the system phase locks. This means that a stable periodic solution arises, in which both oscillators run at some composite frequency, with their phase difference kept constant. The composite frequency need not be constant. In fact, it may periodically oscillate about a constant average value.

2. However, the coupling may also suppress the oscillations, with the resulting system having a stable steady state. This even if none of the component oscillators has a stable steady state. This is oscillator death. It can happen not only for coupled pairs of oscillators, but also for chains of oscillators with coupling to the nearest neighbors.

On the other hand, we note that it is also possible to produce an oscillating system, with a stable oscillation, by coupling non-oscillating systems (e.g., the coupling of excitable systems can do this).

2 Notes: first order equation with a periodic right hand side
You will be asked to justify the statements below in one of the problems. They are stated here because these results are used/needed in the answers to some of the other problems.
Consider the equation
\[ \dot{\phi} = 1 - \kappa \sin \phi, \quad \text{where} \ 0 < \kappa < 1. \] (2.1)
Since \( \dot{\phi} \geq 1 - \kappa > 0 \), \( \phi \) is monotone increasing. It can be shown that:

1. There is a constant \( 0 < \mu < 1 \), and a function \( \Phi = \Phi(\zeta) \) — periodic of period \( 2\pi \) — such that any solution to (2.1) has the form
\[ \phi = \mu (t - t_0) + \Phi(\mu (t - t_0)), \] (2.2)
where \( t_0 \) is a constant and \( \Phi(0) = 0 \). It follows that \( \sin(\phi) \) is periodic in \( t \), with period \( T = \frac{2\pi}{\mu} \)

2. The period-average \( M \) for \( \sin(\phi) \) is given by
\[ M = \text{average}(\sin \phi) = \frac{1 - \mu}{\kappa} > 0. \] (2.3)

3. Let \( \phi_* \) be the solution to (2.1) defined by \( \phi_*(0) = 0 \) — i.e.: set \( t_0 = 0 \) in (2.2). Then
\[ \Theta(\mu t) = \int_0^t (\sin(\phi_*(s)) - M) \, ds = -\frac{1}{\kappa} \Phi(\mu t), \] (2.4)
where \( \Theta \) is defined by the first equality.

4. Assume that \( 0 < \kappa \ll 1 \). Then a Poincaré-Lindstedt expansion yields
\[ \phi_* = \mu t - \kappa (1 - \cos(\mu t)) + O(\kappa^2) \quad \text{and} \quad \mu = 1 - \frac{1}{2} \kappa^2 + O(\kappa^4). \] (2.5)
It follows that \( T = 2\pi + \pi \kappa^2 + O(\kappa^4) \) and \( M = \frac{1}{2} \kappa + O(\kappa^3) \).

5. Assume that \( 0 < 1 - \kappa \ll 1 \). Then
\[ \mu = O(\sqrt{T - \kappa}) \] (2.6)
3 Attracting Lines in the Phase Plane #01

3.1 Statement: Attracting Lines in the Phase Plane #01

You may have observed the frequent appearance of special curves along which solutions tend to bunch up. Sometimes these are associated with the stable and/or unstable manifolds of certain critical points, and sometimes they are not. Below is a particular example, where you are expected to justify the special line with a scaling/asymptotic kind of analysis (coupled with an appropriate qualitative argument on the phase plane). The problem is aimed at testing that you can use these sort of tools properly, so do it using them. If you think of another way of doing it, and you want to show off, then I will look at it and consider it for extra credit — but this can backfire if this “other way” contains some grave conceptual error (thus do it only if you are certain).

Consider the damped pendulum equation:

\[
\frac{d^2\theta}{dt^2} + a \frac{d\theta}{dt} + \sin \theta = 0, \tag{3.1}
\]

where \(a \gg 1\). Show that ALL the trajectories in the phase plane \((\theta, \dot{\theta})\) end up, after a brief transient period, following the curve:

\[
\frac{d\theta}{dt} \approx -\frac{1}{a} \sin \theta. \tag{3.2}
\]

Remark 3.1 (Warnings)

a. Do not fall into the trap of just showing that \(\dot{\theta} + \frac{1}{a} \sin \theta\) is small! Both terms here are small, so showing this provides no new information. What you must show is that this expression is much smaller than either term, for example: \(\dot{\theta} + \frac{1}{a} \sin \theta = O(a^{-2})\). In fact, it is possible to show that \(\dot{\theta} + \frac{1}{a} \sin \theta = O(a^{-3})\).

b. Do not fall into the trap of arguing that, just because some derivative is multiplied by a small parameter, you can neglect it — this is not always true.³ The qualitative argument in the phase plane is the step that will allow you to justify something like this.

c. Show that all the trajectories (eventually) satisfy (3.2), not just that there are some that do.

3.2 Answer: Attracting Lines in the Phase Plane #01

The key idea in solving this problem is to scale the variables properly — notice that the equation is already non-dimensional.

A large damping coefficient \(a \gg 1\) means that the system “does not like” having a large (or even \(O(1)\)) velocity. Any velocity that is not small to begin with, will be rapidly damped (leading to large accelerations) by the balance

\[
\frac{d^2\theta}{dt^2} + a \frac{d\theta}{dt} = -\sin \theta = O(1). \tag{3.3}
\]

In other words, because \(\sin \theta\) is always an \(O(1)\) term, if any of the other terms is large, then both must be large and must cancel each other to leave an \(O(1)\) remainder only. This balance implies damping on a time scale \(\Delta t = O(a^{-1})\), which is thus the time scale for any transient period.

After the transient period described in the prior paragraph, the velocity will be small and the motion rather slow. It is this second stage the problem statement is asking us to look at. Thus it seems obvious that we should re-scale time to look at the motion in the slow time scale. Namely: introduce \(t = \tau/\delta\), where \(\tau\) is the re-scaled time and \(0 < \delta \ll 1\) (with \(\delta\) to be found). The idea is that the time scale over which things happen is \(\Delta \tau = O(1)\) — equivalently: \(\Delta t = O(\delta^{-1})\), so things happen slowly.

The question is now: How do we determine \(\delta\)? Again, we look at equation (3.1) and ask in which ways can the various terms balance (each balance determines a possible time scale).

² Say, in the phase plane plots with the problem set answers, or if you played with the MatLab scripts in the course toolkit.
³ Examples where this fails are wide-spread in applications. They are not a mathematical curiosity.
The balance of the first two terms determines the damping time scale mentioned above. It is not important after the transient period.

The balance of the first and last terms determines the time scale used when nondimensionalizing equation (3.1) — i.e.: it corresponds to $\Delta t = O(1)$ for significant changes. This balance is inconsistent, because: if the time derivatives are $O(1)$, then nothing can balance the middle term, which would then be the only large term in the equation.

By elimination, the only possible balance outside the transient period is that of the second and last terms. This yields $\delta = \frac{1}{a}$.

Thus, define $t = a\tau$, so that the equation becomes

$$\frac{d^2\theta}{d\tau^2} + a^2 \left( \frac{d\theta}{d\tau} + \sin \theta \right) = 0,$$

or

$$\begin{cases} \frac{d\theta}{d\tau} = u, \\ \frac{du}{d\tau} = -a^2 (u + \sin \theta). \end{cases}$$

(3.4)

The behavior of this last system is easy to visualize in the phase plane (see figure 3.1). We have:

- For $u > -\sin \theta$. $\frac{du}{d\tau} \ll -1$ and $\frac{d\theta}{d\tau} = O(1)$. Orbits “zip” down towards the curve $u = -\sin \theta$ (nearly) vertically.

- For $u < -\sin \theta$. $\frac{du}{d\tau} \gg +1$ and $\frac{d\theta}{d\tau} = O(1)$. Orbits “zip” up towards the curve $u = -\sin \theta$ (nearly) vertically.

Figure 3.1: Sketch of the orbits for equation (3.4) in the phase plane, for $a \gg 1$. After a very brief transient period, the orbits approach the curve $u = -\sin \theta$, which they follow within a distance $O(a^{-2})$ — moving away from the saddles and approaching the nodes. The phase plane portrait is periodic in the $\theta$ direction, with period $2\pi$.

Thus the orbits are very quickly attracted to a narrow region (of $O(a^{-2})$ thickness,\(^4\) in this scaling), where they must stay, following the (approximately) 1-D evolution:

$$\frac{d\theta}{d\tau} = u = -\sin \theta + O(a^{-2}).$$

Equivalently:

$$\frac{d\theta}{dt} = -\frac{1}{a} \sin \theta + O(a^{-3}).$$

(3.5)

This is exactly what the problem statement requires be shown.

\(^4\)In order for the “vertical” motion of the solutions in the $(u, \theta)$ phase plane (given by $du/d\tau$) not to dominate over the “horizontal” motion (given by $d\theta/d\tau$), it must be that $a^2 (u + \sin \theta)$ is no larger than $O(1)$.
Remark 3.2 Let us now show two other approaches to this problem, one of which falls into the pitfalls mentioned in remark 3.1, while the other works.

Other approach I — this does NOT work.

We can write equation (3.1) as the following system in the phase plane:

\[
\frac{d\theta}{dt} = u, \quad \text{and} \quad \frac{du}{dt} = -a \left( u + \frac{1}{a} \sin \theta \right).
\]

Then, using an argument similar to the one used above to conclude (3.5), we obtain

\[u + \frac{1}{a} \sin \theta = O(a^{-1}).\]

But this is not good enough, as pointed out in remark 3.1. The reason this approach does not work is the failure to do any re-scaling in the problem.

Other approach II — this works.

Introduce \( v = a \frac{d\theta}{dt} \) and write equation (3.1) as the following system in the phase plane:

\[
\frac{d\theta}{dt} = \frac{1}{a} v, \quad \text{and} \quad \frac{dv}{dt} = -a \left( v + \sin \theta \right).
\]

Then, using an argument similar to the one used above to conclude (3.5), we obtain

\[v + \sin \theta = O(a^{-2}),\]

which is the same as (3.5). Notice that the reasoning leading to the last equation is: for the vertical motion not to dominate over the horizontal one (in the phase plane) it must be that \( \frac{dv}{dt} \) is no larger that \( O(a^{-1}) \), given the equation for \( \frac{d\theta}{dt} \).

♣

Remark 3.3 The formula

\[
\frac{d^2 \theta}{dt^2} \approx -\frac{1}{a} \sin \theta
\]

that the solutions must satisfy (after a transient period) is an approximation, whose error is \( O(a^{-3}) \) — as we pointed out earlier. Can we improve upon this?

The answer is yes. Generally, consider an equation of the form

\[
\frac{d^2 x}{dt^2} + a \frac{dx}{dt} + f(x) = 0, \tag{3.6}
\]

where \( a \gg 1 \) and \( f \) is smooth. Then the same arguments we used for (3.1) show that

\[
\frac{dx}{dt} = -\frac{1}{a} f(x) + O(a^{-3}). \tag{3.7}
\]

Let us try the expansion:

\[
\frac{dx}{dt} = -\frac{1}{a} f(x) + \frac{1}{a^3} g_1(x) + \frac{1}{a^5} g_2(x) + \ldots \tag{3.8}
\]

where \( g_1, g_2, \text{ etc.} \), are functions to be found. It is then clear that

\[
\frac{d^2 x}{dt^2} = \left( -\frac{1}{a} f'(x) + \frac{1}{a^3} g'_1(x) + \frac{1}{a^5} g'_2(x) + \ldots \right) \frac{dx}{dt} + \ldots
\]

\[
= \left( -\frac{1}{2a^2} f''(x) - \frac{1}{a^4} f(x) g_1(x) + \frac{1}{a^6} \left( \frac{1}{2} g_1^2(x) - f(x) g_2(x) \right) + \ldots \right) , \tag{3.9}
\]
where the primes denote derivatives with respect to $x$.

Now, substitute (3.8) and (3.9) into equation (3.6), and equate the coefficients of equal powers of $a$. This yields

\[ g_1 = -\left(\frac{1}{2}f^2\right)' = -ff', \]
\[ g_2 = (f g_1)' = -(f^2 f')' = -\left(\frac{1}{3}f^3\right)'', \]
\[ g_3 = \left(f g_2 - \frac{1}{2}g_1^2\right)'', \]

and so on.

\[ \blacklozenge \]

4 Coupled oscillators #01 (derive phase equations)

4.1 Statement: Coupled oscillators #01

In this problem we present an example of the process described in § 1.1, and consider the coupling of two oscillators with a stable, and strongly attracting, limit cycle each. The oscillators are very simple, with trivial equations in polar coordinates. This simplifies the analysis enormously, but the principles illustrated here are valid for the coupling of more generic oscillators.

Consider the following equations for two coupled oscillators

\[ \dot{x}_j = -\omega_j y_j + \lambda_j \left(R_j^2 - x_j^2 - y_j^2\right)x_j + F_j(x_1, y_1, x_2, y_2), \]
\[ \dot{y}_j = \omega_j x_j + \lambda_j \left(R_j^2 - x_j^2 - y_j^2\right)y_j + G_j(x_1, y_1, x_2, y_2), \]

where $j = 1$ or $j = 2$, and

(a) $\omega_j > 0$, $R_j > 0$ and $\lambda_j > 0$, are constants, with $\lambda_j \gg 1$,

(b) $F_j$ and $G_j$ are some functions — these are the coupling terms.

Using the fact that $\lambda_j \gg 1$, write reduced equations for the two phases $\theta_1$ and $\theta_2$, defined by $x_j = r_j \cos \theta_j$ and $y_j = r_j \sin \theta_j$, where $r_j = \sqrt{x_j^2 + y_j^2}$. In particular, consider the following cases

1. What form do the reduced equations take when the $y_j F_j$ and $x_j G_j$ are only functions of the variables $\eta = x_1 x_2 + y_1 y_2$ and $\xi = y_1 x_2 - x_1 y_2$.

2. What form do the reduced equations take when $G_1 = G_2 = 0$, $F_1 = -\alpha \frac{R_1}{R_2} x_2$, and $F_2 = -\beta \frac{R_2}{R_1} x_1$ — where $\alpha$ and $\beta$ are constants.

Hint. Write the equations in polar coordinates.\(^5\) Then consider what happens in a neighborhood of the limit cycles for the two oscillators when de-coupled — i.e.: $r_j$ not too far from $R_j$. In this context, argue\(^6\) that the dependence on the radial variables can be made trivial.

---

\(^5\) Recall that $r_j \dot{r}_j = x_j \dot{x}_j + y_j \dot{y}_j$ and $r_j^2 \dot{\theta}_j = x_j \dot{y}_j - y_j \dot{x}_j$.

\(^6\) Use arguments similar to the one introduced to describe relaxation oscillations, e.g.: for the van der Pol equation. Another example occurs when justifying that inertial terms can be neglected in the limit of a large viscosity.
4.2 Answer: Coupled oscillators #01

In polar coordinates, \( x_j = r_j \cos \theta_j \) and \( y_j = r_j \sin \theta_j \), the equations take the form

\[
r_j = \lambda_j r_j (R_j^2 - r_j^2) + F_j \cos \theta_j + G_j \sin \theta_j \quad \text{and} \quad \dot{\theta}_j = \omega_j + \frac{1}{r_j} (G_j \cos \theta_j - F_j \sin \theta_j).
\]

(4.3)

Because \( \lambda_j \gg 1 \), for \( r_j \) not too far from \( R_j \), the term \( \lambda_j r_j (R_j^2 - r_j^2) \) dominates the radial time evolution — and drives \( r_j \) towards \( R_j \). It is only when \( r_j - R_j = O(\lambda_j^{-1}) \) that the other terms in the radial equation become significant. Hence the radial time evolution leads to \( r_j \approx R_j \), after which the evolution of the phases, to leading order, reduces to

\[
\dot{\theta}_1 = \omega_1 + K_1(\theta_1, \theta_2) \quad \text{and} \quad \dot{\theta}_2 = \omega_2 + K_2(\theta_1, \theta_2),
\]

(4.4)

where \( \text{K}_j = \frac{1}{R_j^2} \left( \begin{array}{cc} G_j & R_j^2 \cos \theta_1 \sin \theta_1 \cos \theta_2 \sin \theta_2 \\ -F_j & R_j^2 \cos \theta_1 \sin \theta_1 \cos \theta_2 \sin \theta_2 \end{array} \right) \). 

In particular, for the cases in items 1 and 2, we have:

1a. \( K_j = K_j(\theta_1 - \theta_2) \), since \( \eta = R_1 R_2 \cos (\theta_1 - \theta_2) \) and \( \xi = R_1 R_2 \sin (\theta_1 - \theta_2) \).

2a. The equations are: \( \dot{\theta}_1 = \omega_1 + \alpha \sin \theta_1 \cos \theta_2 \) and \( \dot{\theta}_2 = \omega_2 + \beta \sin \theta_2 \cos \theta_1 \).

Note that the equations in (4.3) are singular at \( r_j = 0 \) if \( G_j \) and \( F_j \) do not vanish there — i.e.: if (4.1–4.2) does not have a critical point at the origin. This does not affect the arguments above.

---

5 Friends on a circular track run in opposite directions

5.1 Statement: Friends on a circular track run in opposite directions

In § 8.6 of his book (Nonlinear Dynamics and Chaos) Strogatz introduces the following model equations in the torus

\[
\dot{\theta}_1 = \omega_1 - \kappa_1 \sin (\theta_1 - \theta_2) \quad \text{and} \quad \dot{\theta}_2 = \omega_2 + \kappa_2 \sin (\theta_1 - \theta_2),
\]

(5.1)

where: (a) \( \theta_1, \theta_2 \) are the phases of two oscillators, (b) \( \omega_1, \omega_2 > 0 \) are their natural frequencies, and (c) \( \kappa_1, \kappa_2 > 0 \) are the coupling constants. Then he gives a simple interpretation\(^7\) for the equations as modeling two friends jogging in a circular track of radius \( R \), where \( (\theta_1, \theta_2) \) are their (angular) positions on the track, \( (R \omega_1, R \omega_2) \) are their preferred running speeds, and the coupling models their desire to run together. In this interpretation (5.1) corresponds to the two friends running along the track in the same direction (counter-clockwise). Imagine now a situation where they are running in opposite directions along the track.\(^8\) Then the equations have to be modified to

\[
\dot{\theta}_1 = \omega_1 - \kappa_1 \sin (\theta_1 - \theta_2) \quad \text{and} \quad \dot{\theta}_2 = -\omega_2 + \kappa_2 \sin (\theta_1 - \theta_2).
\]

(5.2)

Equivalently, with \( \varphi_1 = \theta_1 \) and \( \varphi_2 = -\theta_2 \),

\[
\dot{\varphi}_1 = \omega_1 - \kappa_1 \sin (\varphi_1 + \varphi_2) \quad \text{and} \quad \dot{\varphi}_2 = \omega_2 - \kappa_2 \sin (\varphi_1 + \varphi_2).
\]

(5.3)

The motivation given above for (5.2) is not-too-serious, but these equations — in the form given in (5.3) — are a particular example of the type of equations that result when very stable limit cycle oscillators are coupled — e.g., see the problem Coupled oscillators #01.

---

\(^7\) These equations appear in other contexts, as mentioned by Strogatz. Also see § 1.

\(^8\) Coach’s orders, because otherwise they run together, talk, and do not train properly.
Study the behavior of the system in (5.2). In particular: Find all the bifurcations. Is there phase locking?

Note: since (5.2) is a system in the torus, bifurcations that do not appear in the plane might occur, look carefully!

Notation and units: To simplify the answer, assume that the time unit is such that $\omega_1 + \omega_2 = 1$. In addition, use the notation $\kappa = \kappa_1 + \kappa_2$— note $\kappa > 0$.

Hint. Use the results in § 2, “Notes: first order equation with a periodic right hand side”. Do not try to pin-point the various regimes and bifurcations using explicit formulas involving $\omega_1, \omega_2, \kappa_1$ and $\kappa_2$ only — you will also need either $M = M(\kappa)$ or $\mu = \mu(\kappa)$, defined in § 2.

5.2 Answer: Friends on a circular track run in opposite directions

The phase difference $\phi = \theta_1 - \theta_2$ satisfies the equation

$$\dot{\phi} = 1 - \kappa \sin \phi,$$

which is exactly the equation in (2.1). Two cases arise

1. Case $\kappa < 1$. Then we can use the results in § 2. In particular, note that

$$\begin{align*}
\dot{\theta}_1 &= \omega_1 - \kappa_1 \sin \phi = \omega_1 + \frac{\kappa_1}{\kappa} (\dot{\phi} - 1) = \omega_1 - \kappa_1 \frac{1 - \mu}{\kappa} + \frac{\kappa_1}{\kappa} (\dot{\phi} - \mu) \\
&= \omega_1 - \kappa_1 M + \frac{\kappa_1}{\kappa} \frac{d}{dt} \Phi(\mu (t - t_0)). \quad \text{[use equations (2.2–2.3)]}
\end{align*}$$

with a similar result for $\theta_2$. It follows that we can write

$$\begin{align*}
\theta_1 &= \Omega_1 (t - t_0) + \frac{\kappa_1}{\kappa} \Phi(\mu (t - t_0)) + \theta_1(t_0), \\
\theta_2 &= -\Omega_2 (t - t_0) - \frac{\kappa_2}{\kappa} \Phi(\mu (t - t_0)) + \theta_2(t_0),
\end{align*}$$

(5.5)

where $\Omega_1 = \omega_1 - \kappa_1 M < \omega_1$ and $\Omega_2 = \omega_2 - \kappa_2 M < \omega_2$. Notice that $\Omega_1 + \Omega_2 = \mu$, since $\omega_1 + \omega_2 = 1$ and $(\kappa_1 + \kappa_2) M = \kappa M = 1 - \mu$ [see equation (2.3)].

The following sub-cases apply

1a. Both $\Omega_1, \Omega_2 > 0$. Then the oscillators run more-or-less independently of each other, with frequencies (lowered by the interaction) that “wobble” periodically about their means $\Omega_1$ and $\Omega_2$. Since $\Omega_2 = \mu - \Omega_1$, we can think of the solution as a function of the arguments $\chi_1 = \Omega_1 t$ and $\chi_2 = \mu t$ (2π-periodic in each argument). It follows that: The solutions are quasi-periodic, with two periods: $T = 2 \pi/\mu$ and $T_1 = 2 \pi/\Omega_1$ — except if $T/T_1$ is rational, when the solutions are periodic (see remark 5.4).

Question: do these situations actually happen? Answer: yes.

Example #1: Take $\omega_1 = \omega_2 = \frac{1}{2}$ and $\kappa_1 = \kappa_2$. Then $\Omega_1 = \Omega_2 = \frac{1}{4} (1 - \kappa M) = \frac{1}{4} \mu > 0$. A small perturbation of this will keep $\Omega_1, \Omega_2 > 0$, and yield $T/T_1 \approx \frac{1}{2}$ irrational, or rational = quotient of arbitrarily large co-primes.

Example #2: If $\kappa$ is small, $M$ is also small — see equation (2.5). Thus $\Omega_j \approx \omega_j > 0$ for $j = 1, 2$.

1b. Either $\Omega_1 > 0 > \Omega_2$ or $\Omega_1 > 0 > \Omega_2$. This sub-case is quite similar to the one in item 1a, except that one of the two oscillators is running backwards because of the effect of the coupling. The coupling is an “attractive” force between the $\theta_j$’s, and (in this sub-case) one of the oscillators manages to “drag” the other along its own direction, but without managing to actually lock. In fact, not even “average-phase lock” can be achieved within this sub-case: $\Omega_1 = -\Omega_2$ cannot occur, since $\Omega_1 + \Omega_2 = \mu > 0$

Question: do these situations actually happen? Answer: yes. $M$ is determined by $\kappa$. Therefore, keeping $\kappa$ constant, and taking $\omega_j$ (or $\omega_j$) small enough produces the situation.

For the jogging friends in a track, the situation in this sub-case is absurd. It means that one of the runners slows down so much that he/she starts to run backwards!

1c. Either $\Omega_1 = 0$ or $\Omega_2 = 0$. This is a bifurcation threshold, across which one of the oscillators reverses direction. In terms of the torus this means that the orbit switches from winding (around one of the holes) in one direction to the other.
This is a bifurcation which is very peculiar to the torus, since it requires holes that allow a reversal in winding directions.

Note that $\Omega_1 = 0$ does not mean that the first oscillator stops! Look at (5.5), the term involving $\Phi$ is still there! What happens in this case is that $\theta_1$ tries to catch up to $\theta_2$, does not quite make it and it reverses direction and heads towards $\theta_2$ going the other way, overshoots, turns, and the cycle starts away. Sort of what your dog would do if you both were in a circular track, you on a motorcycle running fast along the track, and the dog trying to be with you. The same remarks apply if $\Omega_2 = 0$.

1d. Note that $\Omega_1, \Omega_2 < 0$ cannot happen because $\Omega_1 + \Omega_2 = \mu > 0$.

2. Case $\kappa > 1$. Then the equation for $\phi$ in (5.4) has two fixed points, one unstable (say $\phi_u$) and another stable (say $\phi_s$), with the stable one a global attractor. Thus, as $t \to \infty$, $\phi \to \phi_s$. Plugging this into (5.2), and solving, yields:

$$\theta_1 = \Omega t + \theta_1(0) \quad \text{and} \quad \theta_2 = \Omega t + \theta_2(0),$$

(5.6)

where $\Omega = \omega_1 - \kappa_1 \sin \phi_u = \omega_1 - \kappa_1/\kappa = -\omega_2 + \kappa_2/\kappa = -\omega_2 + \kappa_2 \sin \phi_u$. Then

2a. $\Omega > 0$, phase lock. The first oscillator “wins”, and drags along the second oscillator, making it run backwards relative to its preferred direction (at the price of going slower).

2b. $\Omega < 0$, phase lock. The second oscillator “wins”, and drags along the first oscillator, making it run backwards relative to its preferred direction (at the price of going slower).

2c. $\Omega = 0$, oscillator death. The two oscillators fight each other to a standstill, and nothing moves. Bifurcation transition between the cases in items 2a and 2b.

For the jogging friends in a track, the situation in this case is absurd. It means that one of the runners slows down so much that he/she starts to run backwards!

3. Case $\kappa = 1$. Bifurcation transition between the cases with quasi-periodic solutions with variable frequencies ($\kappa < 1$), and the cases with constant equal frequencies ($\kappa > 1$, phase lock or oscillator death).

Remark 5.4 In case a, near any point at which $T/T_1$ is rational, there are arbitrarily close points where $T/T_1$ is irrational, and where the solutions are qualitatively different. But, these are not bifurcation points! There is no qualitative change as one “crosses the point”, it is the behavior at the point itself that is “singular”. By contrast, note that if $T/T_1$ is irrational, then in a small neighborhood one only finds other irrational points, or rational numbers which are quotients of very large co-primes — hence they correspond to very, very long periods. There is no qualitative discontinuity in the behavior as one approaches a point where $T/T_1$ is irrational.

6 Index for a linearly degenerate node

6.1 Statement: Index for a linearly degenerate node

Consider a phase plane system

$$\dot{x} = f(x, y) \quad \text{and} \quad \dot{y} = g(x, y),$$

(6.1)

where $f$ and $g$ are smooth functions of all of its arguments. Assume that:

1. The origin is a critical point. That is $f(0, 0) = g(0, 0) = 0$. 

2. The origin is a linearly degenerate node. That is: if \( A \) is the 2 \( \times \) 2 matrix corresponding to the linearized system near the critical point at the origin, then \( A \) has two equal, real and nonzero, eigenvalues. Equivalently, if \( \tau = \text{tr}(A) \) and \( \Delta = \text{det}(A) \), then \( 0 < \Delta = \tau^2/4 \).

Because \( \Delta \neq 0 \), the origin is an isolated critical point (inverse function theorem), and has an index associated with it. Let this index be \( I_0 \). Your task here is to calculate \( I_0 \).

Note. Degenerate nodes are structurally unstable. Hence you cannot calculate their index by simply calculating the index for the linearized system, you have to do a slightly more sophisticated calculation. Interestingly, even though the actual phase plane portrait for a degenerate node cannot be ascertained from the linearized system alone (structural instability), you need no nonlinear information to calculate the index!

Hint:

3. Write the system (6.1) in vector form
\[
\dot{Y} = F(Y), \quad \text{where } Y = (x, y)^T \text{ and } F = (f, g)^T.
\]
where \( Y = (x, y)^T \) and \( F = (f, g)^T \).

4. Assume that the matrix \( A \) has the form
\[
A = \begin{pmatrix} \mu & a \\ 0 & \mu \end{pmatrix},
\]
where \( \mu \neq 0 \) is the double eigenvalue, and \( a \) is some constant. There is no loss of generality here: any matrix with a double eigenvalue can be reduced to this form by an appropriate choice of coordinates.

5. Consider now the one parameter family of systems
\[
\dot{Y} = F(Y) + \epsilon \sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Y, \quad 0 \leq \epsilon < \sqrt{\frac{a^2}{4} + \mu^2} - |a| \frac{1}{2},
\]
where \( \sigma = \text{sign}(a) \) if \( a \neq 0 \), and \( \sigma = 1 \) if \( a = 0 \). For these systems the origin is a critical point, with the linearization matrix
\[
B = \begin{pmatrix} \mu & a + \epsilon \sigma \\ \epsilon \sigma & \mu \end{pmatrix}. \quad \text{In particular } \det(B) = \mu^2 - \epsilon (|a| + \epsilon) > 0,
\]
so that the origin is an isolated critical point (inverse function theorem).

Let now \( I = I(\epsilon) \) be the index of the critical point at the origin for (6.4). Calculate \( I(\epsilon) \) for \( \epsilon > 0 \), and use the properties of the index to get \( I(0) = I_0 \).

6.2 Answer: Index for a linearly degenerate node

The eigenvalues for \( B \) are
\[
\lambda = \mu \pm \sqrt{\epsilon (|a| + \epsilon)},
\]
both real and with the same sign — since \( \det(B) > 0 \). It follows that, for \( \epsilon > 0 \), the origin in equation (6.4) is a node. Hence \( I(\epsilon) = 1 \) for \( \epsilon > 0 \). However, (6.4) depends continuously on \( \epsilon \). Thus \( I(\epsilon) \) is a continuous function. We conclude that \( I(0) = I_0 = 1 \).
7 Phase Plane Surgery #01

7.1 Statement: Phase Plane Surgery #01

Can a smooth vector field exist in the plane such that:

— The critical points are $P_1 = (-2,0)$, $P_2 = (0,0)$ and $P_3 = (2,0)$.

— All the critical points are spirals.

— The circles with radii: $R_1 = 1$ centered at $P_1$, $R_2 = 4$ centered at $P_2$, and $R_3 = 1$ centered at $P_3$, are orbits.

Would your answer change if $P_2$ is a saddle?

In either case, if your answer is yes, sketch the way the orbits might look in an example satisfying the criteria above.

Challenge question: In either case, if your answer is yes, can you give an actual example (i.e.: write the vector field explicitly) that gives you a phase portrait with the same qualitative features (the closed orbits need not be circles for this).

7.2 Answer: Phase Plane Surgery #01

The answer to the first question in this problem is: NO.

The reason is that the circle of radius $R_2$ centered at $P_2$ would be an orbit enclosing all three critical points. Thus the sum of the indexes of the critical points would have to be one, which would contradict the fact that the critical points are all supposed to be spirals.

If, on the other hand, $P_2$ is a saddle, then a vector field with the given properties exists.

We will show this by actually constructing one. In our construction the closed orbits doing the job are not be circles, but this is a minor point: The method used to construct the example can be adapted (at the price of much more algebra in the answer) to obtain closed orbits that are circles — or ellipses or any other closed curves one may desire.

Let $V$ be the potential defined by $V = -2x^2 + \frac{1}{4}x^4$. For this potential

— $x = 0$ is a local maximum — with $V(0) = 0$.

— $x = \pm 2$ are local minimums — with $V(-2) = V(2) = -4$.

— Everywhere else $\frac{dV}{dx} \neq 0$.

Define $E = E(x, y)$ by

$$E = \frac{1}{2}y^2 + V(x). \quad (7.1)$$

Then

1. $\nabla E = (E_x, E_y) = \left(\frac{dV}{dx}, y\right)$ vanishes at the points $P_1$, $P_2$, and $P_3$, only.

2. $P_1$ and $P_3$ are local minimums of the surface $z = E(x, y)$, while $P_2$ is a saddle.

3. The level curves $E = c$ = constant (with $-4 < c < 0$) have two components: one enclosing (only) $P_1$ and the other enclosing (only) $P_3$.

4. The level curves $E = c$ = constant (with $0 < c$) have only one component, which encloses all the $P_j$’s and all the level curves with $-4 < E < 0$. 
Consider now the phase plane system given by:

\[
\begin{align*}
\frac{dx}{dt} &= -E_y - f(E) E_x, \\
\frac{dy}{dt} &= E_x - f(E) E_y,
\end{align*}
\]

where \( f = f(E) \) is some function to be defined later, and the superscript \( \perp \) indicates rotation by \( \pi/2 \) of a vector in the plane.

**Remark 7.5 (Motivation).** The flow provided by the system of equations in (7.2) has two components. The first one, given by \((\nabla E)^\perp\), induces a flow along the level lines of the surface \( z = E(x, y) \) — for this component alone the orbits would be the level curves of \( E \). The second component, given by \( f \nabla E \), induces a flow normal to the level lines of \( E \) — for this component alone the orbits would correspond to the path of water flowing down/up\(^9\) the gravity gradient on the surface \( z = E \).

Thus, for the system in (7.2), the level curves of \( E \) corresponding to zeros of \( f(E) \) are orbits. On the other hand, when \( f \) is small, the orbits will (approximately) track the level curves of \( E \) — slowly drifting up or down the gradient of \( E \), depending on the sign of \( f \). Thus, given the properties of \( E \) — itemized below equation (7.1), it should be clear that: by judiciously choosing \( f = f(E) \), we can obtain a system with the desired properties.

A clarification regarding orbits and level lines: when above we say “the orbits would be level lines of \( E \)”, or similar, it should be clear that this applies for each component of \( E = \) constant, if \( E = \) constant is made up by more than one curve — e.g.: when \(-4 < E < 0 \). The case \( E = 0 \) (saddle level) has three components, one where \( x > 0 \), another where \( x < 0 \), and the critical point \( P_2 \). Finally, the case \( E = -4 \) has two components: the critical points \( P_1 \) and \( P_3 \).

\[ \star \]

Below we fill in the details of the intuitive arguments in the remark.

For the system in (7.2), we have

\[
\begin{align*}
\dot{x} - \dot{y} &= -(1 + f^2) E_x, \\
\dot{y} + \dot{x} &= -(1 + f^2) E_y,
\end{align*}
\]

from which it follows that the critical points of (7.2) are exactly the \( P_j \)'s — since these are the only points where \( \nabla E \) vanishes. It is also clear that, along orbits:

\[
\frac{dE}{dt} = \dot{x} E_x + \dot{y} E_y = - (E_x^2 + E_y^2) f(E).
\]

Hence:

Any level curve \( E = E_0 = \) constant, such that \( f(E_0) = 0 \), is an orbit.

(7.5)

Now take

\[
f = \epsilon (E^2 - 4),
\]

(7.6)

where \( \epsilon > 0 \) is an arbitrary constant. With this choice for \( f = f(E) \), note that:

5. The critical points \( P_1 = (-2, 0) \) and \( P_3 = (2, 0) \) are stable spirals if \( \epsilon < \sqrt{2}/49 \approx 0.0673 \ldots \), or stable improper nodes if \( \epsilon = \sqrt{2}/49 \), or stable nodes if \( \sqrt{2}/49 < \epsilon \).

This follows from the \( 2 \times 2 \) matrix of the linearized equations near the critical points, both given by:

\[
A = \begin{bmatrix} -96 \epsilon & -1 \\ 8 & -12 \epsilon \end{bmatrix}.
\]

6. The critical point \( P_2 = (0, 0) \) is always a saddle.

This follows from the \( 2 \times 2 \) matrix of the linearized equations near the critical point, given by:

\[
A = \begin{bmatrix} -16 \epsilon & -1 \\ -4 & 4 \epsilon \end{bmatrix}.
\]

\[ \text{Down if } f > 0; \text{ up if } f < 0. \]
7. For any orbit that is not a critical point, (7.4) yields

\[
\text{As } t \to \infty \quad E \to 2 \quad \text{or} \quad E \to -4.
\]
\[
\text{As } t \to -\infty \quad E \to -2 \quad \text{or} \quad E \to \infty.
\]

(7.7)

A sketch of a proof is at the end of this problem answer.

**Proviso:** If an orbit has \( E \to \infty \) in the direction of increasing \( t \), then the limit is achieved for a finite value of \( t \).

The other values (\( E = \pm 2 \) and \( E = -4 \)) require either \( t \to \infty \) or \( t \to -\infty \).

Thus, provided that \( 0 < \epsilon < \sqrt{2}/49 \approx 0.0673 \ldots \), we have:

8. The two level curves \( E = -2 \) are unstable limit cycles enclosing the critical points \( P_1 \) and \( P_3 \).

9. The level curve \( E = 2 \) is a stable limit cycle enclosing all the critical points.

10. The critical points \( P_1 \) and \( P_3 \) are stable spiral points, while \( P_2 \) is a saddle point.

![Figure 7.1: Phase portrait for the system in (7.2), with \( \epsilon = 0.025 \) in (7.6). The two unstable limit cycles are given by the components of the level curve \( E = -2 \), and the single stable limit cycle is given by the level curve \( E = 2 \).](image)

Figure 7.1 shows the phase portrait of the system for the choice \( \epsilon = 1/40 \). The degree of stability of the limit cycles, and of the critical points \( P_1 \) and \( P_3 \), is controlled by \( \epsilon \) — as should be obvious from (7.4). As \( \epsilon \) grows, the limit cycles become more stable (resp. unstable), and at \( \epsilon = \sqrt{2}/49 \) the critical points \( P_1 \) and \( P_3 \) switch from spirals to nodes. Unless \( \epsilon \) is fairly small, the orbits move away (towards) the limit cycles so fast that they appear as the spokes of bicycle tires near them.

It should be obvious that the shape and location of the limit cycles can be changed — by replacing \( E \) by some other function with different level curves. It is possible to get the limit cycles to have almost any desired shape (at the price of having to manufacture a complicated\(^{10}\) function \( E \)).

**Sketch of the proof of (7.7).** Consider an orbit that is not a critical point. Then \( E_x^2 + E_y^2 > 0 \) everywhere on the orbit — although \( E_x^2 + E_y^2 \) may approach zero if the orbit approaches a critical point as either \( t \to \infty \), or \( t \to -\infty \).

\(^{10}\) An explicit expression for \( E \) might not even be possible. But the surface \( z = E(x, y) \) need not be “geometrically” more complicated than the one given by (7.1).
Then (7.4) shows that $E$ is increasing for $-2 < E < 2$, and decreasing for $|E| > 2$. Note that $E$ is restricted to the range $-4 \leq E < \infty$.

8 Example of a reversible system that is not conservative

8.1 Statement: Example of a reversible system that is not conservative

Give an example of a reversible system that is not conservative.

Hint. Remember that a phase plane system $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y), \quad (8.1)$
is reversible if, for example, $f$ is odd and $g$ is even in $y$ — that is: $f(x, -y) = -f(x, y)$ and $g(x, -y) = g(x, y)$. In this case the change $t \rightarrow -t$ and $y \rightarrow -y$ leaves the system invariant.

In addition, we know that conservative systems cannot have sinks or sources. Now, ask yourself: what systems have exactly the opposite property [almost every critical point is either a source or a sink]. Then produce a system of this kind, with $f$ odd and $g$ even.

8.2 Answer: Example of a reversible system that is not conservative

Consider a gradient system: $\dot{x} = -V_x$ and $\dot{y} = -V_y$, where $V = V(x, y)$ is some given function. For systems of this type, any local minimum of $V$ is a sink, and any local maximum is a source. Now take $V$ odd in $y$; then the system is also reversible. Hence any gradient system where $V$ is odd in $y$, and where $V$ has either a local maximum or a local minimum, provides the required example. One is $V = y (1 - y^2)/(1 + x^2)$, which has a local minimum at $(x, y) = (0, -1/\sqrt{3})$, and a local maximum at $(x, y) = (0, 1/\sqrt{3})$.

9 Problem 14.09.22 (Attracting and Liapunov stable)

9.1 Statement for problem 14.09.22

Recall the definitions for the various types of stability that concern critical points:

Let $x^*$ be a fixed point of the system $\dot{x} = f(x)$. Then:

1. $x^*$ is attracting if there is a $\delta > 0$ such that $\lim_{t \rightarrow \infty} = x^*$ whenever $\|x(0) - x^*\| < \delta$. That is: any trajectory that starts within $\delta$ of $x^*$ eventually converges to $x^*$. Note that trajectories that start nearby $x^*$ need not stay close in the short run, but must approach $x^*$ in the long run.

2. $x^*$ is Liapunov stable if for each $\epsilon > 0$, there is a $\delta > 0$ such that $\|x(t) - x^*\| < \epsilon$ for $t > 0$, whenever $\|x(0) - x^*\| < \delta$. Thus, trajectories that start within $\delta$ of $x^*$ stay within $\epsilon$ of $x^*$ for all $t > 0$.

In contrast with attracting, Liapunov stability requires nearby trajectories to remain close for all $t > 0$.

3. $x^*$ is asymptotically stable if it is both attracting and Liapunov stable.

4. $x^*$ is repeller if there are $\epsilon > 0$ and $\delta > 0$ such that if $0 < \|x(0) - x^*\| < \delta$, after some critical time $\|x(t) - x^*\| > \epsilon$ applies (i.e., for $t > t_c$). Repellers are a special kind of unstable critical points.

For each of the following systems, decide whether the origin is attracting but not Liapunov stable, Liapunov stable, asymptotically stable, repeller, or unstable but not a repeller.
Example of a reversible system that is not conservative.

\[ \dot{x} = 2y \quad \text{and} \quad \dot{y} = -3x. \]

b) \[ \dot{x} = y \cos(x^2 + y^2) \quad \text{and} \quad \dot{y} = -x \cos(x^2 + y^2). \]

c) \[ \dot{x} = -x \quad \text{and} \quad \dot{y} = -|y|y. \]

d) \[ \dot{x} = 2xy \quad \text{and} \quad \dot{y} = y^2 - x^2. \quad \text{Hint: what happens along } x = 0? \]

e) \[ \dot{x} = x - 2y x^2 - 4y^3 \quad \text{and} \quad \dot{y} = y + x^3 + 2xy^2. \]

f) \[ \dot{x} = y \quad \text{and} \quad \dot{y} = x. \]

g) Finally, consider the critical point \((x, y) = (1, 0)\), for the system

\[ \dot{x} = (1 - r^2) x - (1 - \frac{x}{r}) y \quad \text{and} \quad \dot{y} = (1 - r^2) y + (1 - \frac{y}{r}) x, \quad (9.1) \]

defined in the “punctured” plane \(r = \sqrt{x^2 + y^2} > 0\). \text{Hint: write the equations in polar coordinates.}

Additional hints. In some cases you can get the answer by finding a function \(J = J(x, y)\) with a local minimum at the origin such that \(\frac{dJ}{dt} > 0\) along trajectories — or maybe one such \(\frac{dJ}{dt} < 0\), or maybe one such \(\frac{dJ}{dt} = 0\). In other cases look for special trajectories that either leave, or approach, the origin.

9.2 Answer for problem 14.09.22

These are the answers, for each system:

a) It is easy to check that \(J = 3x^2 + 2y^2\) is constant along the trajectories.
   The origin is a center, thus \textit{Liapunov stable} — it is, in fact, a \textit{linear, stable, node}.

b) It is easy to check that \(J = x^2 + y^2\) is constant along the trajectories.
   The origin is a center, thus \textit{Liapunov stable}.

c) In terms of the initial values, the solution for \(t > 0\) is \(x = x_0 e^{-t}\) and \(y = y_0/(1 + |y_0|t)\).
   The origin is \textit{asymptotically stable} — it is, in fact, a \textit{nonlinear, stable, node}.

d) The first equation shows that, if \(x\) vanishes anywhere, then it vanishes everywhere — i.e., \(x = 0\) is an invariant curve. The other equation then reduces to \(\dot{y} = y^2\). Thus \(x = 0\) and \(y > 0\) is an orbit leaving the origin, and \(x = 0\) and \(y < 0\) is an orbit approaching the origin.
   The origin is \textit{unstable, but not a repeller}.
   Note: for this system it can be shown that all the trajectories that have \(x \neq 0\) somewhere (thus everywhere, why?) approach the origin as \(t \to \infty\).

e) It is easy to check that \(J = x^2 + 2y^2\) satisfies \(\frac{dJ}{dt} = 2\dot{J}\) along the trajectories.
   The origin is a \textit{repeller}.
   Note: in fact, the origin is a nonlinear, unstable, spiral. This follows from the fact that the equations yield, for the polar angle \(\theta\), \(\dot{\theta} = J/r^2\) — where \(r^2 = x^2 + y^2\).

f) It is easy to check that \(J = x^2 - y^2\) is constant along the trajectories.
   The origin is a saddle (in fact, a linear saddle), thus \textit{unstable, but not a repeller}.

g) In polar coordinates equation (9.1) takes the form

\[ \dot{r} = (1 - r^2) r \quad \text{and} \quad \dot{\theta} = 1 - \cos \theta. \quad (9.2) \]

From this it should be clear that, for any initial data \(0 < r_0\) and \(0 < \theta < 2\pi\), the solution approaches the critical point at \(t \to \infty\). Furthermore, the real positive axis (i.e., \(\theta = 0\)) is an invariant curve, and along it the system reduces to \(\dot{x} = (1 - x^2) x\) — thus, again, the trajectories approach the critical point as \(t \to \infty\). It follows that the \textit{critical point is attracting}.
On the other hand, consider the trajectory \( r \equiv 1 \) and \( 0 < \theta < 2\pi \). This trajectory starts at the critical point at \( t = -\infty \), goes around the unit circle counterclockwise, and arrives back at the critical point at \( t = \infty \). In fact, any solution starting near the critical point with \( y > 0 \), initially moves away from the critical point (as far as a distance \( \approx 2 \)), before returning to the critical point as \( t \to \infty \). Thus the critical point is not Liapunov stable.

## 10 Problem 06.01.08 - Strogatz (Computer generated phase portrait)

### 10.1 Statement for problem 06.01.08

Plot a computer generated phase plane portrait for the van der Pol oscillator

\[
\frac{dx}{dt} = y, \quad \text{and} \quad \frac{dy}{dt} = -x + y(1-x^2). 
\]  

(10.1)

### 10.2 Answer for problem 06.01.08

See figure 10.1.

![van der Pol oscillator: \( x_t = y, \ y_t = -x + (1-x^2)y \).](image)

Figure 10.1: (Problem 06.01.08). Phase plane portrait for the van der Pol oscillator.

## 11 Two limit cycles and enclosed critical points

### 11.1 Statement: Two limit cycles and enclosed critical points

Consider a phase plane system of the form

\[
\frac{dx}{dt} = f(x, y) \quad \text{and} \quad \frac{dy}{dt} = g(x, y),
\]  

(11.1)
where \( f \) and \( g \) are both smooth. Assume that:

A. The circles \( r = 1 \) and \( r = 2 \) are limit cycles.

B. There is a saddle point at \((x, y) = (1.5, 0)\).

Show that there must be at least one more critical point in the annular region \( 1 < r < 2 \) between the two limit cycles.

11.2 Answer: Two limit cycles and enclosed critical points

The index of a closed orbit is one, and it is equal to the sum of the indexes of all the enclosed critical points. Thus the indexes of all the critical points in \( r < 1 \) add up to one, and the same must be true of the indexes for the critical points in \( r < 2 \). It follows that the sum of the indexes of the critical points in \( 1 < r < 2 \) must vanish. Since the saddle at \((x, y) = (1.5, 0)\) has index -1, there must be at least another critical point in \( 1 < r < 2 \).

THE END.