Hopf bifurcations using two timing and complex notation
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November 24, 2018

Abstract
In these notes we illustrate how the use of complex notation can dramatically simplify calculations for (at least some) problems. In particular, we show a Hopf bifurcation calculation. You should compare these notes with the section Hopf bifurcation for second order scalar equations in the Hopf bifurcation notes.

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1 Problem formulation

Consider a system in the plane dependent on a parameter $r$

$$\frac{d\vec{u}}{dt} = \vec{f}(\vec{u}, r), \quad \vec{f} \text{ is smooth.} \tag{1.1}$$

Assume now that (1.1) has an isolated critical point that, at some value $r = r_c$, changes stability: from a stable to an unstable spiral (or the reverse). Without loss of generality, we will assume that the critical point is the origin, and that $r_c = 0$. Then, for $\vec{u}$ and $r$ small we can write

$$\frac{d\vec{u}}{dt} = A\vec{u} + \vec{I}_2(\vec{u}) + \vec{I}_3(\vec{u}) + rB\vec{u} + O(\epsilon^4, r^2 \epsilon), \tag{1.2}$$

where $\epsilon = \|\vec{u}\|$, $A$ and $B$ are $2 \times 2$ matrices, $\vec{I}_2$ involves only quadratic terms in $\vec{u}$, and $\vec{I}_3$ involves only cubic terms in $\vec{u}$. Furthermore, because the origin is a center for $r = 0$, we know that $A$ has a (complex) eigenvector $\vec{v}$, with eigenvalue $i \mu$ (where $\mu > 0$). That is:

$$A\vec{v} = i \mu \vec{v} \iff A\vec{v}_1 = \mu \vec{v}_2 \quad \text{and} \quad A\vec{v}_2 = -\mu \vec{v}_1, \tag{1.3}$$

where $\vec{v} = \vec{v}_1 - i \vec{v}_2$ ($\vec{v}_j$ real) and we can write any vector as a linear combination of the $\vec{v}_j$. In particular

$$\vec{u} = x \vec{v}_1 + y \vec{v}_2, \quad \text{so that} \quad A\vec{u} = \mu (-y \vec{v}_1 + x \vec{v}_2). \tag{1.4}$$

Note then that, in terms of the complex number $z = x + iy$, the action by $A$ is equivalent to multiplication by $i \mu$. Hence we can write (1.2) in the equivalent complex form

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1 Because $\vec{u} = 0$ is assumed to be a critical point for all $r$ small, there are no $O(r^n)$ terms.
\[ \dot{z} = i \mu z + r (a_1 z + a_2 \bar{z}) + (b_1 z^2 + b_2 z \bar{z} + b_3 \bar{z}^2) + (c_1 z^3 + c_2 z^2 \bar{z} + c_3 z \bar{z}^2 + c_4 \bar{z}^3) + O(\epsilon^4, r^2 \epsilon), \quad (1.5) \]

where (i) \( \bar{z} \) denotes the complex conjugate and (ii) the \( a_j, b_j, \) and \( c_j \) are complex constants (for a generic system, they are unrestricted).

## 2 Two times expansion – no quadratic terms

Consider the situation where the quadratic terms in (1.5) vanish. Then we assume \( r = \nu \epsilon^2, \nu = \pm 1, \) so that the linear perturbation term and the cubic nonlinearity have the same size, and propose a two-times expansion of the form

\[ z = \epsilon z_1(t, \tau) + \epsilon^3 z_3(t, \tau) + \ldots \quad (2.1) \]

where \( \tau = \epsilon^2 t \) and the dependence on \( t \) is periodic — it should be easy to see that no \( O(\epsilon^2) \) terms are needed. Then \( z_0 = \mathcal{A}(\tau) e^{i\mu t} \) and the \( O(\epsilon^3) \) yield

\[ \dot{z}_3 - i \mu z_3 = -\dot{\mathcal{A}} e^{i\mu t} + \nu a_1 \mathcal{A} e^{i\mu t} + c_2 |\mathcal{A}|^2 \mathcal{A} e^{i\mu t} + \text{NRT}, \quad (2.2) \]

where the Non Resonant Terms (NRT) have \( t \)-dependences proportional to \( e^{-i\mu t}, e^{-3i\mu t}, \) and \( e^{3i\mu t} \). Suppressing resonant terms then yields

\[ \frac{d\mathcal{A}}{d\tau} = \nu a_1 \mathcal{A} + c_2 |\mathcal{A}|^2 \mathcal{A}. \quad (2.3) \]

Write now \( \mathcal{A} = \rho e^{i\phi}, \) with \( \rho \) and \( \phi \) real. Then (2.3) becomes

\[ \frac{d\rho}{d\tau} = (\nu \text{Re}(a_1) + c_2 \rho^2) \rho \quad \text{and} \quad \frac{d\phi}{d\tau} = \nu \text{Im}(a_1) + \text{Im}(c_2) \rho^2. \quad (2.4) \]

Note now that \( \text{Re}(a_1) \neq 0, \) because of the assumption that the origin switches stability and the expansion in (1.5) — which yields linearized equations \( \dot{z} = i \mu z + r (a_1 z + a_2 \bar{z}) + O(r^2 \epsilon). \) Thus, the condition for a Hopf bifurcation is \( \text{Re}(c_2) \neq 0. \) Then

1. If \( \text{Re}(a_1)/\text{Re}(c_2) = \kappa^2 > 0, \) a limit cycle with radius \( \rho = \kappa \) arises for \( \nu = -1. \) The bifurcation is supercritical (soft) if \( \text{Re}(a_1) < 0, \) and subcritical (hard) if \( \text{Re}(a_1) > 0. \)

2. If \( \text{Re}(a_1)/\text{Re}(c_2) = -\kappa^2 < 0, \) a limit cycle with radius \( \rho = \kappa \) arises for \( \nu = 1. \) The bifurcation is supercritical (soft) if \( \text{Re}(a_1) > 0, \) and subcritical (hard) if \( \text{Re}(a_1) < 0. \)

3. In either case, the second equation in (2.4) indicates that the angular frequency for the limit cycle, up to the order considered, is \( \omega = \mu + r \text{Im}(a_1) + \epsilon^2 \text{Im}(c_2) \kappa^2. \)

Recall that for a supercritical bifurcation the limit cycle that arises is stable, while it is unstable for a subcritical bifurcation.

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\[ ^2 \text{We leave it as an exercise to consider the general case.} \]

\[ ^3 \text{They are needed when the quadratic terms in (1.5) do not vanish.} \]