

# Continuum Media and Elasticity.

Rodolfo R. Rosales, Department of Mathematics,

Massachusetts Institute of Technology, Cambridge, Massachusetts, MA 02139

April 12, 2010

## Contents

<b>1 Elasticity: Set up and Forces.</b>	<b>2</b>
Definition of an elastic solid . . . . .	2
Non pre-stressed elastic solid . . . . .	2
Description of an elastic solid: Lagrangian and Eulerian coordinates . . . . .	3
Displacement field . . . . .	3
Local description: infinitesimal neighborhood . . . . .	3
Deformation decomposition: rotation, shear, and compression/expansion . . . . .	4
Example: small deformations . . . . .	4
The strain tensor . . . . .	4
Tracking angles and lengths. Scalar products and the strain tensor . . . . .	5
Transformation properties of the strain tensor . . . . .	5
1.1 The small deformations limit . . . . .	6
Hooke's law . . . . .	6
Stress-strain relationship for isotropic media . . . . .	6
Lamé constants and the shear modulus . . . . .	6
Colinearity of the stress and the strain . . . . .	7
Should the Lamé constants be positive? . . . . .	7
Elastic deformation energy . . . . .	8
The other constants of linear elasticity . . . . .	8
Bulk modulus . . . . .	8
Shear modulus . . . . .	8
Young's modulus . . . . .	8

<i>MIT, (Rosales).</i>	Continuum Media and Elasticity.	2
	Poisson's ratio . . . . .	9

**Problems. 10**

**2 Evolution of the potential energy along particle paths. 10**

2.1 Statement: Evolution of the potential energy along particle paths . . . . .	10
Derive the equations for a continuum. Calculate the evolution along particle paths of the internal energy. Show that angular momentum is conserved . . . . .	10

**3 Uniform strain elasticity solution. 11**

3.1 Statement: Uniform strain elasticity solution . . . . .	11
Construct a special solution of the elasticity equations, corresponding to a uniform, but arbitrary, strain field . . . . .	11

**4 Energy in linear elasticity. 12**

4.1 Statement: Energy in linear elasticity . . . . .	12
Directly verify conservation of energy in linear elasticity. . . . .	12

**1 Elasticity: Set up and Forces.**

**Definition 1.1** *An elastic solid is characterized by the fact that each infinitesimal element in the solid has an equilibrium shape — to which it will return if all the external forces on the element are relaxed. To be more precise: deformations of the shape from the equilibrium one generate forces that oppose the deformations, and the forces depend on the deformations only. The forces are short range (infinitesimal in the continuum limit) so that they are described by a stress tensor.*

**Definition 1.2** *A non pre-stressed elastic solid is an elastic solid such that the whole solid<sup>1</sup> possesses an equilibrium shape — where the stress tensor vanishes.*

**In these notes we deal with non pre-stressed elastic solids only.**

---

<sup>1</sup>Not just each infinitesimal element in it.

*Pre-stressed elastic solids are quite common: it is very hard to obtain truly non pre-stressed materials. Most elastic objects, when all external forces are removed, relax to an equilibrium state where some stresses remain: Each element could relax to a zero stress shape, but their arrangement in space does not allow this simultaneously. However, as long as the deformation scales are much larger than the scale of the “residual” stresses, the non pre-stressed approximation is reasonable.*

Given a non pre-stressed elastic solid, introduce an inertial, cartesian, coordinate system and, for each point  $\vec{x}$  in the solid, let:

$$\vec{s} = \text{coordinates of the point when the solid is at rest and in its equilibrium shape.} \quad (1.1)$$

Then  $\vec{s}$  provides a system of Lagrangian coordinates for the solid — i.e.: a label for each mass point. The solid is completely described, at any time, by the function

$$\vec{x} = \vec{X}(\vec{s}, t) \quad \text{— or its inverse } \vec{s} = \vec{S}(\vec{x}, t) \quad \text{—} \quad (1.2)$$

relating the current and equilibrium positions of each mass point. We can also write

$$\vec{x} = \vec{s} + \vec{u} \quad \text{or} \quad \vec{s} = \vec{x} - \vec{u}, \quad \text{where } \vec{u} = \vec{u}(\vec{x}, t) = \vec{u}(\vec{s}, t) = \text{displacement field.} \quad (1.3)$$

**Remark 1.1** *The definition of  $\vec{s}$  in (1.1) is not unique. Given a set of Lagrangian solid coordinates  $\vec{s}$ , for any  $\mathbf{R} = \text{constant rotation matrix}$  and  $\vec{s}_0 = \text{constant vector}$ ,  $\vec{s}_* = \mathbf{R}\vec{s} + \vec{s}_0$  is also an acceptable set of Lagrangian coordinates. Here we assume that some choice has been made.*

#### LOCAL DESCRIPTION: INFINITESIMAL NEIGHBORHOOD.

Consider an infinitesimal neighborhood of some fixed point, and write

$$\vec{x} = \vec{x}_0 + \delta\vec{x} \quad \text{and} \quad \vec{s} = \vec{s}_0 + \delta\vec{s}. \quad (1.4)$$

Then

$$\delta\vec{x} = \mathbf{G} \delta\vec{s} \quad \text{and} \quad \delta\vec{s} = \mathbf{H} \delta\vec{x}, \quad (1.5)$$

where  $\mathbf{G}$  and  $\mathbf{H}$  are the matrices defined by  $G_{ij} = \frac{\partial x_i}{\partial s_j}$  and  $H_{ij} = \frac{\partial s_i}{\partial x_j}$ , with  $\mathbf{G} = \mathbf{H}^{-1}$ .

We can write

$$\mathbf{H} = d\mathbf{H}_r \mathbf{H}_s, \quad (1.6)$$

where  $d > 0$  is a scalar,  $\mathbf{H}_r$  is a **rotation matrix** (orthogonal with determinant one), and  $\mathbf{H}_s$  is a **symmetric matrix with positive eigenvalues and determinant one**. This follows from the fact that any matrix can be written as the product of a rotation matrix times a symmetric matrix (polar decomposition), and the **assumption** that the solid deformations are the result of a continuous process starting from equilibrium. Proof:

From the polar decomposition we can write  $H = H_r \tilde{H}_s$ , where  $\tilde{H}_s$  is symmetric. But  $H$  is not singular, hence  $H = d H_r H_s$ , where  $d = (\det(\tilde{H}_s))^{1/3}$ , and  $H_s = d^{-1} \tilde{H}_s$ . This formula should apply at every step of the deformation process from equilibrium — hence neither  $d$ , nor any of the eigenvalues of  $H_s$  can cross zero. But, at equilibrium:  $d = 1$  and  $H_r = H_s = 1$ .

The decomposition in (1.6) shows that **all the local deformations of the solid can be reversed by first shearing** (deformation without volume change), **followed by a rotation, and either an expansion (if  $d > 1$ ) or a contraction (if  $d < 1$ )**. Since we can also write  $H = d \tilde{H}_s \tilde{H}_r$ , the order (rotation first) of these operations can be changed.

**Example 1.1 Small deformations.** From (1.3) we can write

$$\mathbf{H} = \mathbf{1} - \mathbf{D}, \quad \text{where} \quad D_{ij} = \frac{\partial u_i}{\partial x_j}. \quad (1.7)$$

Now assume that the deformations are small, namely  $D = O(\epsilon)$ , where  $0 < \epsilon \ll 1$ . Then

$$\mathbf{H} = \mathbf{1} - \alpha - D_a - D_s = (1 - \alpha)(1 - D_a)(1 - D_s) + O(\epsilon^2) = (1 - \alpha) e^{-D_a} e^{-D_s} + O(\epsilon^2), \quad (1.8)$$

where  $\alpha = \text{Tr}(D)$ ,  $D_a = \frac{1}{2}(D - D^T)$  is the anti-symmetric part of  $D$ , and  $D_s = \frac{1}{2}(D + D^T) - \alpha$  is the trace-less symmetric part of  $D$ . Comparing (1.8) with (1.6) shows that

$$d = (1 - \alpha) + O(\epsilon^2), \quad H_r = e^{-D_a} + O(\epsilon^2), \quad \text{and} \quad H_s = e^{-D_s} + O(\epsilon^2). \quad (1.9)$$

## THE STRAIN TENSOR.

The information about the deformations (shape changes) in the solid is encoded in  $d H_s$  in (1.6). The rotations, encoded in  $H_r$ , have no effect on the forces (stresses) generated when the elastic

solid is deformed. Hence, the stress tensor should be a function of  $dH_s$  only. In fact, it is more convenient to write everything in terms of the **strain tensor**

$$\boldsymbol{\sigma} = \frac{1}{2} (\mathbf{1} - \mathbf{H}^T \mathbf{H}), \quad \text{namely: } \sigma_{ij} = \frac{1}{2} \left( \delta_{ij} - \frac{\partial s_\ell}{\partial x_i} \frac{\partial s_\ell}{\partial x_j} \right), \quad (1.10)$$

where we use the repeated index summation convention — note that  $\boldsymbol{\sigma}$  is **a-dimensional**. Clearly  $\boldsymbol{\sigma} = \frac{1}{2} (\mathbf{1} - d^2 H_s^2)$  encodes the same information as  $dH_s$ . Thus, **for an elastic non pre-stressed solid, the stress tensor is a function of the strain tensor**

$$\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{\sigma}), \quad \text{with } \boldsymbol{\tau}(\mathbf{0}) = \mathbf{0}. \quad (1.11)$$

**Remark 1.2** Equation (1.11) applies in the case of an homogeneous elastic solid. **If the solid is not homogeneous, then** .....  $\boldsymbol{\tau} = \boldsymbol{\tau}(\vec{\mathbf{s}}, \boldsymbol{\sigma})$ , with  $\boldsymbol{\tau}(\vec{\mathbf{s}}, \mathbf{0}) \equiv \mathbf{0}$ .

#### TRACKING ANGLES AND LENGTHS. SCALAR PRODUCTS.

Another way to justify (1.11) is as follows: the amount of deformation in an elastic solid can be determined if we know how distances and angles change. This is equivalent to knowing how the scalar product changes. However, it is easy to see that

$$\langle \delta \vec{x}_1, \delta \vec{x}_2 \rangle - \langle \delta \vec{s}_1, \delta \vec{s}_2 \rangle = 2 (\delta x_1)^T \boldsymbol{\sigma} \delta x_2, \quad (1.12)$$

since, from (1.5),  $\delta \vec{s}_j = H \delta \vec{x}_j$ . Thus the strain tensor encodes, precisely, the information on how distances and angles change by the deformation of the solid from equilibrium.

#### TRANSFORMATION PROPERTIES OF THE STRAIN TENSOR.

Consider two cartesian frames related by a **rotation**  $\mathbf{R}$ . That is

$$\vec{\mathbf{x}}_b = \mathbf{R} \vec{\mathbf{x}}_a \quad \text{and} \quad \vec{\mathbf{s}}_b = \mathbf{R} \vec{\mathbf{s}}_a. \quad \text{Then, just as for the stress tensor, } \boldsymbol{\sigma}_b = \mathbf{R} \boldsymbol{\sigma}_a \mathbf{R}^T. \quad (1.13)$$

This justifies the use of the appellation “tensor” for  $\boldsymbol{\sigma}$ . Proof:

As in (1.2), write  $\vec{\mathbf{s}}_b = \vec{S}_b(\vec{\mathbf{x}}_b, t)$ , where the gradient of  $\vec{S}_b$  is  $H_b$  — see (1.5). Hence  $\vec{\mathbf{s}}_a = \mathbf{R}^T \vec{S}_b(\mathbf{R} \vec{\mathbf{x}}_a, t) \implies H_a = \mathbf{R}^T H_b \mathbf{R}$ . Using this in (1.10), (1.13) follows.

## 1.1 The small deformations limit.

First we summarize the formulas obtained earlier, as they become in the case of small deformations.

**Assume that  $D$  in (1.7) is small —  $O(\epsilon)$  with  $0 < \epsilon \ll 1$ , and neglect  $O(\epsilon^2)$  terms.**

**Remark 1.3** *The fact that  $D$  is small does not imply the  $\vec{u}$  is small. A thin flat plate (or a thin rod) can undergo significant lateral displacements while keeping the deformations small.*

We have 
$$H = 1 - D \quad \text{and} \quad G = 1 + D, \quad (1.1.1)$$

as follows from  $G = H^{-1}$  — with  $G$

and  $H$  as defined in (1.5). Hence 
$$\frac{\partial u_i}{\partial s_j} = D_{ij} \quad \text{and} \quad \sigma = \frac{1}{2} (\mathbf{D} + \mathbf{D}^T). \quad (1.1.2)$$

### HOOKE'S LAW.

If  $\sigma$  is small, and  $\tau$  is a smooth function of  $\sigma$  — see (1.11 – 1.12), then we can **approximate  $\tau$  by a linear function of  $\sigma$  — Hooke's law**. In principle, this involves 36 coefficients (because the symmetric tensors  $\tau$  and  $\sigma$  involve 6 independent entries each). However, **if the media is isotropic, the relationship between  $\sigma$  and  $\tau$  has to be invariant under rotations**. From (1.13), and its analog for  $\tau$ , this means that

$$\tau(\mathbf{R} \sigma \mathbf{R}^T) = \mathbf{R} \tau(\sigma) \mathbf{R}^T \quad (1.1.3)$$

for any rotation matrix  $\mathbf{R}$ . This reduces the number of parameters to two,<sup>2</sup> so that

$$\tau = \lambda \text{Tr}(\sigma) \mathbf{1} + 2 \mu \sigma, \quad (1.1.4)$$

where  $\lambda > 0$  and  $\mu > 0$  are the **Lamé constants** —  $\mu$  is the **shear modulus**,  $\sigma$  is as in (1.1.2), and we note that (1.7) yields

$$\text{Tr}(\sigma) = \text{div}(\vec{u}) = \frac{\partial u_i}{\partial x_i}. \quad (1.1.5)$$

**A.** In terms of the inequality  $\lambda > 0$  above: In fact,  $\lambda > -(2/3) \mu$  is the only “hard” restriction, though for most common materials  $\lambda > 0$ . We deal with this, and related issues, in remark 1.5.

---

<sup>2</sup>See remark 1.4.

**B.** For a non-homogeneous solid,  $\lambda = \lambda(\vec{s})$  and  $\mu = \mu(\vec{s})$ .

**Remark 1.4** (1.1.4) *implies stress and strain colinearity:  $\tau$  and  $\sigma$  have the same principal axes.* At every point there is a rotation matrix  $R$  such that both  $R\tau R^T$  and  $R\sigma R^T$  are diagonal. Conversely, **if colinearity is assumed, (1.1.4) follows.** In this case the problem reduces to characterizing all the linear transformations between 3 by 3 diagonal matrices, with the property that the transformations are invariant under permutations of the diagonal elements. The most general linear transformation between 3 by 3 diagonal matrices involves 9 coefficients  $c_{ij}$  — defined by the equations  $d(\tau)_i = c_{ij} d(\sigma)_j$ , where  $d(\tau)_i$  and  $d(\sigma)_j$  are the diagonal elements. Further: there are 6 permutations to enforce invariance under. A little work then shows that it must be  $c_{ij} = a$  if  $i = j$  and  $c_{ij} = b$  if  $i \neq j$ , for some constants  $a$  and  $b$ . This is (1.1.4), with  $a = \lambda + 2\mu$  and  $b = 2\mu$ .

**That colinearity should apply is intuitively “obvious”.** Consider a deformation that involves no rotation — there is no loss of generality in this, because the stress tensor ignores rotations. Then, in (1.6)  $H = d H_s$ , hence  $\sigma = \frac{1}{2}(1 - H^2)$ . Observe now what happens with an infinitesimal cube whose facets are normal to the principal axes of strain: the cube is deformed in such a way that each facet moves normal to itself by some amount determined by the corresponding principal strain. Under these conditions, how can shear stresses (parallel to the facets) arise? Such a situation would not be invariant under rotation, for the shear stresses would determine special directions in space which are not encoded within  $\sigma$ . Hence no such shear stresses can arise, meaning that colinearity applies. Unfortunately, this argument is not 100 % tight. The gaps can be filled, but then the elegance is lost. A better argument leading to (1.1.4) uses the relationship between the stress tensor and the elastic energy stored in the solid.

**Remark 1.5 Should the Lamé constants in (1.1.4) be positive?** This cannot follow from any symmetry arguments, such as the ones in remark 1.4. It (partly) follows from the fact that the stresses generated by an elastic solid oppose the deformations that cause them: in order to deform an elastic solid, energy must be spent.<sup>3</sup> This spent energy then goes into stored elastic deformation

---

<sup>3</sup>The opposite would give rise to very unstable (and rather un-physical) situations, with deformations growing without bound in the absence any applied forcing. Within the linear regime, energy out of nothing!

energy. It can be shown that **the elastic deformation energy  $V$**  (per unit mass) associated with (1.1.4) has the form

$$\begin{aligned}\rho V &= \frac{1}{2} \lambda (\text{Tr}(\boldsymbol{\sigma}))^2 + \mu \text{Tr}(\boldsymbol{\sigma}^T \boldsymbol{\sigma}) \\ &= \frac{1}{2} \lambda \left( \sum_i d_i(\boldsymbol{\sigma}) \right)^2 + \mu \sum_i d_i^2(\boldsymbol{\sigma}),\end{aligned}\tag{1.1.6}$$

where the  $d_i(\boldsymbol{\sigma})$  are the principal values (eigenvalues) of  $\boldsymbol{\sigma}$ , and  $\rho$  is the density. From the arguments above, it should be  $E > 0$  for any non-trivial  $\boldsymbol{\sigma}$ . In other words, the quadratic form in the second line of (1.1.6) should be positive definite. A simple calculation shows that this is equivalent to

$$\mu > 0 \quad \text{and} \quad 3\lambda + 2\mu > 0.\tag{1.1.7}$$

This allows  $\lambda$  non-positive in the range  $-(2/3)\mu < \lambda \leq 0$  — see the Poisson's ratio discussion below.

#### THE CONSTANTS OF LINEAR ELASTICITY.

The **bulk modulus  $B = \lambda + \frac{2}{3}\mu > 0$**  relates the traces of  $\tau$  and  $\sigma$ . Namely, from (1.1.6)

$$\text{Tr}(\boldsymbol{\tau}) = 3B \text{Tr}(\boldsymbol{\sigma}).\tag{1.1.8}$$

However: **(A)** The **pressure** (the shear-less component of the stress) is given by  $p = -\frac{1}{3} \text{Tr}(\boldsymbol{\tau})$ . **(B)** From (1.5) and (1.1.1 – 1.1.2), the **volume change relative to the equilibrium state** is given by  $\Delta \text{Vol} = \det(\boldsymbol{G}) - 1 = \text{div}(\vec{u}) = \text{Tr}(\boldsymbol{\sigma})$ . Hence

$$\mathbf{p} = -\mathbf{B} \Delta \text{Vol}.\tag{1.1.9}$$

By contrast, the **shear modulus  $\mu > 0$**  relates the trace-less part of  $\sigma$  (deformation without volume change; i.e.: shear) with the trace-less part of  $\tau$  (what remains of  $\tau$  after the pressure is removed).

Now consider a situation where the stress is uni-directional,<sup>4</sup> namely: only one principal stress is non-zero. Then the ratio of the non-zero stress to its corresponding strain is the **Young's modulus  $E$** .

<sup>4</sup>Such as it occurs when pulling (along its axis) on a thin rod of the elastic material.



Aligning the coordinate axes with the stress (and strain) tensors principal axes, this leads — using (1.1.4) — to

$$E \sigma_{11} = \tau_{11} = (\lambda + 2\mu) \sigma_{11} + \lambda (\sigma_{22} + \sigma_{33}), \quad (1.1.10)$$

$$0 = \tau_{22} = (\lambda + 2\mu) \sigma_{22} + \lambda (\sigma_{11} + \sigma_{33}), \quad (1.1.11)$$

$$0 = \tau_{33} = (\lambda + 2\mu) \sigma_{33} + \lambda (\sigma_{11} + \sigma_{22}), \quad (1.1.12)$$

where  $\tau_{11}$  is the non-vanishing principal stress. Thus

$$E = \frac{3\lambda + 2\mu}{\lambda + \mu} \mu > 0 \quad \text{and} \quad \sigma_{22} = \sigma_{33} = -\frac{\lambda}{2(\lambda + \mu)} \sigma_{11}. \quad (1.1.13)$$

Finally, the **Poisson's ratio**  $\nu$  is the (negative of) the ratio between the side strains and the main strain in the same uni-axial stress situation defining  $E$ . From (1.1.13)

$$-1 < \nu = \frac{\lambda}{2(\lambda + \mu)} < \frac{1}{2}, \quad (1.1.14)$$

where the inequalities follow from (1.1.7). Clearly: the Poisson ratio is a measure of the transverse contraction (or expansion) when the material is under uni-axial stress. Furthermore:  $\nu > 0$  **corresponds to contraction when the material is under tension and stretched** —  $\tau_{11}$  and  $\sigma_{11}$  both positive, and expansion in the opposite situation. **Most material behave like this**, hence (in general)

$$\nu > 0 \implies \lambda > 0. \quad (1.1.15)$$

However this is a consequence of the “typical” microscopic structure of common materials, and not of some deep physical principle. In fact **man made materials exist (foams) for which  $\nu < 0$**  — **hence  $\lambda < 0$** . This materials have a rather un-intuitive behavior: when a rod made of them is stretched along the axis, its cross-sectional area grows. By the same token, the cross-section decreases when pushing the rod inwards from its ends.

Finally, notice that we can write

$$\nu = \frac{3\gamma - 2}{2(3\gamma + 1)}, \quad \text{where} \quad \gamma = \frac{B}{\mu}. \quad (1.1.16)$$

Hence, materials with  $\nu < 0$  have  $\gamma$  “small”, and resist shear better than compression. On the other hand, materials that resist compression much better than shearing (e.g.: rubber) have  $\gamma$  “large”, hence  $\nu$  close to 0.5. An example of a material with  $\nu$  small — i.e.:  $\gamma$  close to  $2/3$  — is cork.

## Problems.

### 2 Evolution of the potential energy along particle paths.

#### 2.1 Statement: Evolution of the potential energy along particle paths.

Consider a continuum media (solid or liquid) in some region of space, with an inertial cartesian coordinate system  $\vec{x}$ . Let the media have the following fields associated with it:  $\rho = \rho(\vec{x}, t)$  = mass density,  $\vec{v} = \vec{v}(\vec{x}, t)$  = mass flow velocity,  $\tau = \tau(\vec{x}, t)$  = stress tensor,  $\vec{f} = \vec{f}(\vec{x}, t)$  = body forces per unit mass, and  $V = V(\vec{x}, t)$  = internal energy per unit volume. Assume that all these fields are smooth functions of  $(\vec{x}, t)$ , that  $\tau$  is a **symmetric rank two tensor**, and then:

1. **Derive p.d.e.’s that the fields above should satisfy** — using conservation of mass, momentum, and energy.
2. Use the equations in item 1 to **obtain expressions for the flow velocity and internal energy material derivatives**. That is, for  $\frac{D\vec{v}}{Dt} = \vec{v}_t + (\vec{v} \cdot \nabla)\vec{v}$  and  $\frac{DV}{Dt} = V_t + (\vec{v} \cdot \nabla)V$ . Note that in the equation for  $\frac{DV}{Dt}$  the body forces should not appear.
3. **Write the equation for the angular momentum** and show that, given the equations in item 1, it is **satisfied if and only if  $\tau$  is symmetric**.

**Remark 2.1** *The equations above have to be completed with appropriate equations of state. For example, in gas dynamics  $\tau$  is given in terms of the pressure  $p$  and  $\vec{v}$ , while  $V$  is given in terms of  $p$  and  $\rho$ . These notes describe how this is done in elasticity.*

### 3 Uniform strain elasticity solution.

#### 3.1 Statement: Uniform strain elasticity solution.

Consider an homogeneous elastic solid, with equilibrium density  $\rho_0 > 0$ , and let  $H = H(t)$  be a  $3 \times 3$  matrix valued smooth function of time, such that:

$$\mathbf{H} = \mathbf{H}^T > \mathbf{0} \text{ is symmetric and positive definite (the eigenvalues of } H \text{ are positive).} \quad (3.1)$$

Assume now that the elastic solid is described — see equations (1.1) through (1.5) — by

$$\vec{s} = \mathbf{H}(t) \vec{x} \iff \vec{x} = \mathbf{G}(t) \vec{s} \quad (3.2)$$

where  $\mathbf{G} = \mathbf{H}^{-1}$ . This corresponds to the velocity field, strain tensor, and density, given by<sup>5</sup>

$$\vec{v} = \frac{d\mathbf{G}}{dt} \mathbf{H} \vec{x} = -\mathbf{G} \frac{d\mathbf{H}}{dt} \vec{x}, \quad \sigma = \frac{1}{2} (1 - \mathbf{H}^2), \quad \text{and} \quad \rho = \rho_0 \det(\mathbf{H}). \quad (3.3)$$

Since  $\rho$  and  $\sigma$  are functions of time only, **the stress tensor is a function of time only:  $\tau = \tau(t)$ .**

**Remark 3.1** *Since  $H$  can be selected to take any positive definite symmetric matrix value, it should be clear — see (1.6) and (1.10) — that  $\sigma$  can be made to take any possible value that a strain tensor can have.*

These are the **tasks in this problem:**

**A. Show** that the equation for the conservation of mass (see the problem in §2) is satisfied.

**B. Show** that appropriate body forces  $\vec{f}$  can be selected to make the equation for the conservation of momentum (see the problem in §2) apply. Compute  $\vec{f}$ .

**C. Use the equation for the conservation of energy** (see the problem in §2) to derive a relationship between  $\frac{dV}{dt}$  and  $\tau$ , where  $V$  is the elastic potential energy per unit mass in the solid — note that  $V$  is also a function of time only.

**Hints.** *The following facts will be useful: Let  $U = U(t)$  be a matrix valued function, with  $U$  nonsingular. Then, in general,  $\frac{dU}{dt}$  and  $U$  do not commute. Hence.*

---

<sup>5</sup>The flow velocity follows from  $\vec{v} = d\vec{x}/dt$  with  $\vec{s}$  constant. Then use that, since  $\mathbf{G}\mathbf{H} = \mathbf{1}$ ,  $\dot{\mathbf{G}}\mathbf{H} + \mathbf{G}\dot{\mathbf{H}} = \mathbf{0}$ .

1.  $\frac{dU^2}{dt} = \frac{dU}{dt} U + U \frac{dU}{dt}.$
  2.  $\frac{dU^{-1}}{dt} = -U^{-1} \frac{dU}{dt} U^{-1}.$
  3.  $\frac{d(\ln \det U)}{dt} = \text{Tr} \left( U^{-1} \frac{dU}{dt} \right) = \text{Tr} \left( \frac{dU}{dt} U^{-1} \right),$  where  $\text{Tr}$  denotes the trace of a matrix.
- 

## 4 Energy in linear elasticity.

### 4.1 Statement: Energy in linear elasticity.

The equations for isotropic linear elasticity can be written in the form

$$\rho \vec{u}_{tt} = \rho \vec{f} + \text{div}(\tau), \quad (4.1)$$

where  $\rho = \rho(\vec{x})$  is the density,  $\vec{f} = \vec{f}(\vec{x}, t)$  are the body forces,  $\vec{u} = \vec{u}(\vec{x}, t)$  is the displacement vector, and  $\vec{v} = \vec{u}_t$  is the velocity. The strain and stress tensors are given by

$$\sigma_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad (4.2)$$

$$\tau = \lambda \text{div}(\vec{u}) + 2\mu \sigma, \quad (4.3)$$

where  $\lambda > -(2/3)\mu$  and  $\mu > 0$  are the Lamé “constants” — though they may be functions of  $\vec{x}$  if the media is not homogeneous. Finally, the elastic deformation energy per unit mass  $V$  is given by

$$\rho V = \frac{1}{2} \lambda (\text{Tr}(\sigma))^2 + \mu \text{Tr}(\sigma^T \sigma). \quad (4.4)$$

**Write the appropriate equation for the conservation of energy, and verify that it is satisfied.**

---

**The End.**