# Stress Tensor



$\boxed{\text{Note: Use the matrix notation in p 5 from the word go}}$

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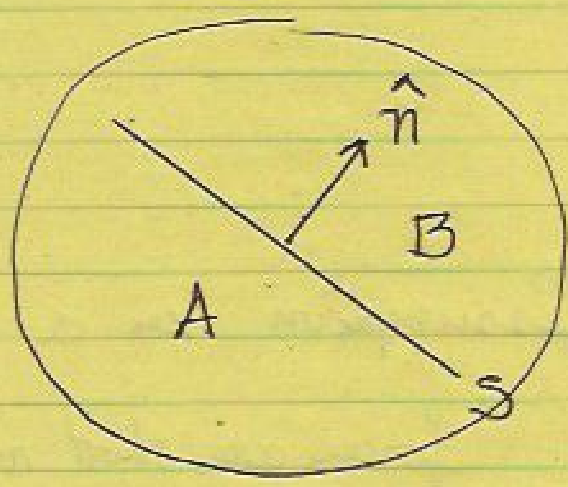
Let $\vec{x}$ be an inertial, cartesian, coordinate system, and consider some substance/material under stress




(1) $\rightarrow$

Define, for any element of surface $d\vec{S}$ with unit normal $\hat{n}$ (see figure)

$$\left.\begin{array}{l}\vec{F} = \vec{F}(\hat{n}, \vec{x}) = \text{force per unit} \\ \qquad \text{area across the surface} \\ \qquad \text{by } \text{side B on side A}\end{array}\right] \quad (2)$$

Since $\vec{F}$ = linear momentum flux, by the same argument as in the problem "Why is the flux a vector"

$\vec{F}$ is given by a tensor Namely

$$\vec{F} = \hat{n} \cdot \tau \quad \text{or} \quad F_n = \hat{n}_m \tau_{mn} \qquad (3)$$

where $\tau = \tau(\vec{x})$ is the stress tensor

Notation (4)

a) We use the repeated index summation convention

b) The dot denotes tensor contraction, so that $\hat{n} \cdot \tau = \hat{n}_j \tau_{ji}$ and

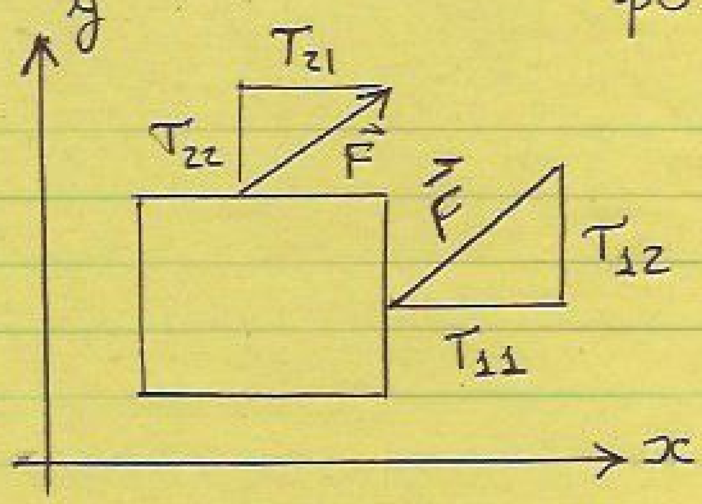
$\tau \cdot \hat{n} = \tau_{ji} \hat{n}_i$

c) $\nabla$ is the vector $\frac{\partial}{\partial x_j}$

d) $\operatorname{div} \tau = (\tau_{ji})_{x_j}$

e) $\tau^{+}$ is the transpose of $\tau$

Two-D picture illustrating the meaning of $\tau$




## $\tau$ is a symmetric tensor (5)

$$\tau_{ij} = \tau_{ji}$$

Proof Consider a sphere of radius R centered at some arbitrary point $\vec{x}_0$. Denote the sphere by $\Omega$ and its boundary by $\partial\Omega$. The torque on the sphere is then

$$\vec{T} = \int_{\partial\Omega} (\hat{n}\cdot\tau)\times(\vec{x}-\vec{x}_0)\, dS =$$

$$= \int_{\Omega} \mathrm{div}\left[\tau \times (x-\vec{x}_0)\right] dV$$

where

$$\operatorname{div}\left[ \tau \times (\underbrace{\vec{x} - \vec{x}_0}_{\vec{z}}) \right] = (\operatorname{div} \tau) \times (\vec{x} - \vec{x}_0) + \{ \tau_{ij} \epsilon_{lji} \}$$

since $(\tau \times \vec{z})_{il} = \tau_{ij} \epsilon_{ljm} z_m$

However, in order to avoid unbounded angular accelerations†, $\vec{T}$ must behave like $O(R^4)$ as $R \to \infty$.

Hence $\tau_{ij} \epsilon_{lji} = 0 \quad \forall l \Rightarrow \tau_{ij} = \tau_{ji}$

† If $\vec{a}$ is the acceleration and $\rho$ the mass density, then

$$\int_{\Omega} \rho \vec{a} \times (\vec{x} - \vec{x}_0) \, dV = \vec{T}$$

ignoring body forces and assuming $\Omega$ is not changing

# Transformation Properties

Let the two cartesian coordinates $\vec{x}_a$ & $\vec{x}_b$ be related by the orthogonal transf. $\underline{\underline{T}}$. Namely

$$\boxed{\vec{x}_b = T\vec{x}_a \text{ , Then } \tau_b = T\tau_a T^+} \quad (6)$$

Proof Use matrix notation, with vectors being column vectors and $\tau$ the matrix $\{\tau_{ij}\}$. Then (3) $\Longleftrightarrow$

$$\vec{F} = \tau^+\hat{n} = \tau\hat{n} \quad (7)$$

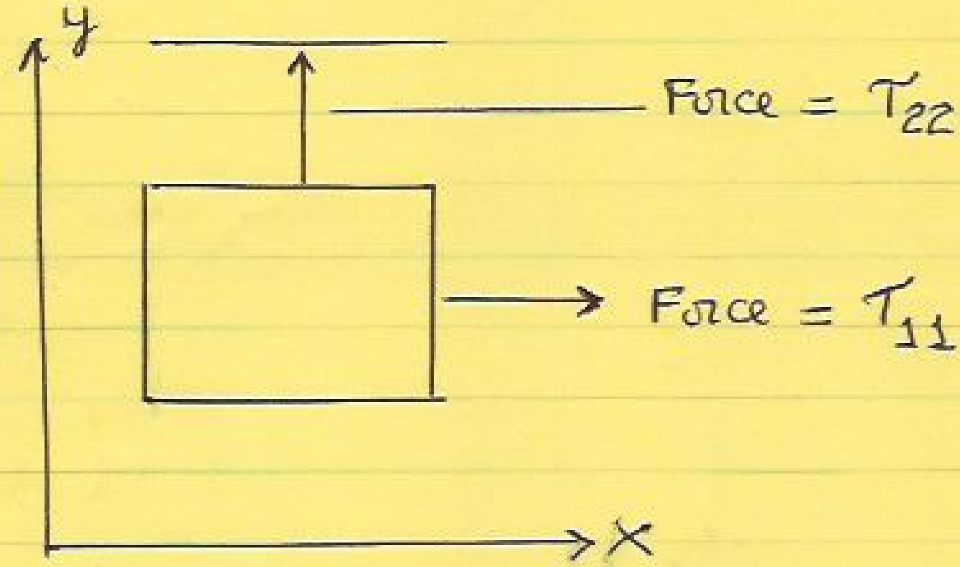
$\tau = \tau^+$ from (5)

Now $\vec{F}_b = T\vec{F}_a$ , $\hat{n}_b = T\hat{n}_a \Rightarrow$

$$\vec{F}_b = T\vec{F}_a = T\tau_a\hat{n}_a = \underline{\underline{T\tau_a T^+}}\hat{n}_b$$

## Principal Axes & physical meaning

At each point a rotation can be applied that makes $\tau$ diagonal. The coordinate axis where this happen are the principal axes. Then




(8)

Squeeze and pull. No shear along sides of elementary cube of sides $dx$

## Pressure - shear decomposition of stress tensor

Let $p = -\frac{1}{3} \, \text{Trace} \, T = T_{nn}$

Then $\quad T = -pI + T^* \qquad (9)$

Where $T^*$ is symmetric with zero trace (hence the sum of the 3 principal forces by $T^*$ is zero)

$p$ is the pressure. If $T^* = 0$ then the stress is normal to any force, and independent of direction.

The response to pressure only ~~[illegible]~~ by an isotropic material will be a volume change. In elastic, isotropic, materials $p$ is caused by volume changes (same as for fluids) →

On the other hand, if $p = 0$, the response of an isotropic material will be shape changes (with the same volume) In fluids $T^*$ is produced by shear (viscous forces) while in elastic media it results from shear deformations