1 Amplitude-phase evolution for the modulation equations

1.1 Statement: Amplitude-phase evolution for the modulation equations

Consider an arbitrary dispersive system: $u_t + i \Omega(-i \partial_x) u = 0$, where $\omega = \Omega(k)$ is a real valued (for $k$ real) smooth function. Assume the equation is in $\epsilon$-dimensional variables.

As shown elsewhere, solutions of the form $u \sim A e^{i \theta}$ (where $A = A(\chi, \tau)$, $\theta = \frac{1}{\epsilon} \Theta(\chi, \tau)$, $\chi = \epsilon x$, $\tau = \epsilon t$, and $0 < \epsilon \ll 1$) lead to the equations

$$\omega = \Omega(k) \quad \text{and} \quad A_t + c_g A_x + \frac{1}{2} (c_g)_x A = 0,$$

(1.1)

where: $\omega = \omega(\chi, \tau) = -\Theta_x = -\theta_t$, $k = k(\chi, \tau) = \Theta_x = \theta_x$, and $c_g = \Omega'(k)$. Write $A = \rho e^{i \phi}$ (polar representation) and find the equations that $\rho$ and $\phi$ satisfy. What is the equation for $\rho^2$?
1.2 Answer: Amplitude-phase evolution for the modulation equations

Substituting \( A = \rho e^{i\phi} \) into (1.1), and collecting the real and imaginary parts, yields:

\[
\rho\tau + c_g \rho_x + \frac{1}{2} (c_g)_x \rho = 0 \quad \text{and} \quad \phi_x + c_g \phi_x = 0. \tag{1.2}
\]

Multiplying the first equation by \( 2 \rho \) yields:

\[
(\rho^2)_\tau + (c_g \rho^2)_x = 0. \tag{1.3}
\]

We conclude that: (i) The phase \( \phi \) is constant along points moving at the group speed, \( \frac{dx}{d\tau} = c_g \); and (ii) The square of the amplitude is a conserved density that flows at the group speed.

2 Amplitude modulation with frozen carrier frequency

2.1 Statement: Amplitude modulation with frozen carrier frequency

Consider a dispersive equation in n-D of the form

\[
u_t + \Omega(\varepsilon \nabla) \mathbf{u} = 0, \tag{2.1}
\]

where: (i) \( u = u(\mathbf{x}, t) \) is a scalar function, (ii) \( \nabla \) is the gradient operator in n-D, and (iii) \( \omega = \Omega(\mathbf{k}) \) is a smooth, real valued (scalar), of the wave-number vector \( \mathbf{k} \in \mathbb{R}^n \). \footnote{If you want to, you may assume that \( \Omega \) is a polynomial: a finite sum \( \Omega = \sum a_{i_1 i_2 \ldots i_n} k_1^{i_1} k_2^{i_2} \ldots k_n^{i_n} \), for some coefficients \( a_{i_1 i_2 \ldots i_n} \). However, this is not actually necessary.}

This equation is dispersive, with dispersion relation \( \omega = \Omega(\mathbf{k}) \).

Consider now solutions of the form

\[
u \sim A(\mathbf{x}, \tau) e^{i\theta}, \tag{2.2}
\]

with \( \mathbf{x} = \varepsilon \mathbf{x}, \tau = \varepsilon t, \) and \( \theta = \mathbf{k}_0 \cdot \mathbf{x} - \omega_0 t \), where: (i) \( 0 < \varepsilon \ll 1 \) is a constant, (ii) \( \mathbf{k}_0 \) is a constant wave-number vector, and (iii) \( \omega_0 = \Omega(\mathbf{k}_0) \).

Derive the leading order equation satisfied by \( A \).

**Hint.** Think of \( \mathbf{u} \) as a function of \( \mathbf{x} \) and \( \mathbf{x}' \), with \( \mathbf{x} \) and \( \mathbf{x}' \) independent variables (same for \( t \) and \( \tau \)). Then the equation can be written in the form

\[
u_t + \epsilon \nu_x + i \Omega(\varepsilon \nabla_1 - \varepsilon \nabla_2) \mathbf{u} = 0, \tag{2.3}
\]

where \( \nabla_1 \) is the gradient operator for \( \mathbf{x} \) and \( \nabla_2 \) is the gradient operator for \( \mathbf{x}' \). Now expand the operator \( \Omega(\varepsilon \nabla_1 - \varepsilon \nabla_2) \) in powers of \( \varepsilon \), and notice that evaluating functions of \( -i \nabla_1 \) is easy — just replace \( -i \nabla_1 \) by \( \mathbf{k}_0 \) (do you see why this?).

**Remark 2.1** Consider a function, \( f = f(x) \), evaluated on an operator (or matrix) \( f = f(A) \). \footnote{This can certainly be done when \( f \) is a polynomial. But this is not the only case: For example, if \( f \) is analytic, we can use Cauchy’s theorem to write \( f(A) = \frac{1}{2\pi i} \int (z - A)^{-1} f(z) dz \), where the contour of integration includes the spectrum of \( A \).}

When doing this we need to keep in mind that, because operators do not commute, the “regular” rules of calculus may not apply. For example: (i) Generally we can write \( \frac{d}{dt} f(A(t)) = f'(A(t)) \frac{dA}{dt} \) only if \( A \) and \( \frac{dA}{dt} \) commute. (ii) Generally we can “Taylor” expand \( f(A + B) \), for \( B \) small, only if \( A \) and \( B \) commute.

2.2 Answer: Amplitude modulation with frozen carrier frequency

We follow the hint, and write

\[
i \Omega(\varepsilon \nabla_1 - \varepsilon \nabla_2) = i \Omega(\varepsilon \nabla_1) + \varepsilon \nabla_1 (i \Omega(\varepsilon \nabla_1) \cdot \nabla_2 + O(\varepsilon^2)). \tag{2.4}
\]

Using this in (2.3), with \( u \) as in (2.2), yields

\[
\left( -i \omega_0 A + \epsilon A_x + i \Omega(\mathbf{k}_0) A + \varepsilon \mathbf{c}_g(\mathbf{k}_0) \cdot \nabla_2 A + O(\varepsilon^2) \right) e^{i\theta} = 0. \tag{2.5}
\]
Since $\omega_0 = \Omega(\vec{0})$, we obtain (at leading order) $A_\tau + \vec{c}_g(\vec{0}) \cdot \nabla_2 A = 0$. \hfill (2.6)

We can do this because $\nabla_1$ and $\nabla_2$ commute. See remark 2.1.

## 3 Example of modulation equations for a slowly varying media

### 3.1 Statement: Example of modulation equations for a slowly varying media

The purpose of this is to illustrate how the tools developed in the class can be used to deal with variable media, in the limit where the wave-length is much shorter than the scale over which the media varies. Thus consider the example:

\[ u_{tt} - (c^2 u_x)_x + m^2 u = 0 \] \hfill (3.1)

where: (i) A-dimensional variables where the wave-length and wave-period are $O(1)$. (ii) $c = c(\chi) > 0$ and $m = m(\chi) > 0$, where $\chi = \epsilon x$ and $0 < \epsilon \ll 1$ is the ratio between the wave-length and the scale over which the media varies.

Consider now a solution of the form $u \sim A e^{i \theta}$, where: $A = A(\chi, \tau), \theta = \frac{1}{\epsilon} \Theta(\chi, \tau)$, and $\tau = \epsilon t$. Derive the equations that govern the evolution of $k = \theta_x = \Theta_x$, $\omega = -\theta_t = -\Theta_\tau$, and $A$. Furthermore, introduce the polar representation $A = \rho e^{i \phi}$ and interpret the meaning of the equations in terms of the group speed and the average energy: (i) How is the evolution of $\phi$ related to the group speed? Hint: what happens along the curves defined by $\frac{dx}{d\tau} = \vec{c}_g$. (ii) Show that the average wave energy $\mathcal{E}$ is conserved, with flux $\mathcal{F} = \vec{c}_g \mathcal{E}$. Hint: the equation for the conservation of energy is

\[ \left( \frac{1}{2} u_t^2 + \frac{1}{2} c^2 u_x^2 + \frac{1}{2} m^2 u^2 \right)_t + (-c^2 u_x u_t)_x = 0. \] \hfill (3.2)

Use this to compute the average energy density and flux, and show that $\mathcal{F} = \vec{c}_g \mathcal{E}$. Then show that the equations that you derived yield $\mathcal{E}_\tau + \mathcal{F}_x = 0$. Warning: when computing $\mathcal{E}$ and $\mathcal{F}$ you need to use a real valued solution, that is $u = \text{Re}(A e^{i \theta}) = \rho \cos(\theta + \phi)$. Hint. You will need to use $k_\tau + \omega_\chi = 0$, which follows from the definition of $k$ and $\omega$.

### 3.2 Answer: Example of modulation equations for a slowly varying media

Substituting $A e^{i \theta}$ into the equation yields:

\[ (\omega^2 - c^2 k^2 - m^2) A + i \epsilon (\omega_\tau A + 2 \omega A_\tau + 2 k c^2 A_\chi + (k c^2)_\chi A) + O(\epsilon^2) = 0. \] \hfill (3.3)

Thus:

\[ \omega^2 = c^2 k^2 + m^2 \] \hfill (3.4)

and

\[ \omega_\tau A + 2 \omega A_\tau + 2 k c^2 A_\chi + (k c^2)_\chi A = 0. \] \hfill (3.5)

In terms of the polar representation $A = \rho e^{i \phi}$ the last equation is

\[ (\omega \rho^2)_\tau + (k c^2 \rho^2)_\chi = 0 \quad \text{and} \quad \omega \phi_\tau + k c^2 \phi_\chi = 0. \] \hfill (3.6)

It is easy to see that $c_g = k \omega c_g$. Hence $\phi$ is constant along the curves $\frac{d\chi}{d\tau} = \vec{c}_g$.

Furthermore, (3.4), $k_\tau + \omega_\chi$, and (3.6) imply that

\[ (\omega^2 \rho^2)_\tau + (\omega k c^2 \rho^2)_\chi = 0. \] \hfill (3.7)
On the other hand (3.2) gives $E = \frac{1}{2} \omega^2 \rho^2$, $F = \frac{1}{2} c^2 k \omega^2 = c_g E$. Thus (3.7) is the same as

$$E_x + F_x = 0,$$

(3.8)

with $F = c_g E$.

### 4 Wave modulation equations using Fourier Transforms

#### 4.1 Statement: Wave modulation equations using Fourier Transforms

Consider a dispersive wave system, with (real valued) dispersion function

$$\omega = \Omega(k), \text{ where } -\infty < k < \infty,$$

(4.1)

which you can assume has as many derivatives as you need. Then the general solution to the system (in $-\infty < x < \infty$) can be written using inverse Fourier Transforms

$$u = \int_{-\infty}^{\infty} U(k) e^{i(kx-\omega t)} \, dk,$$

where $\omega = \Omega(k)$,

(4.2)

and $U$ is the Fourier Transform of the initial data

$$U = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x,0) e^{-ikx} \, dx.$$

(4.3)

Assume now that the initial data corresponds to a slowly varying monochromatic wave

$$u(x,0) = \alpha(X) e^{i\theta_0}, \text{ where } \theta_0 = \frac{1}{\epsilon} \Phi(X), \quad X = \epsilon x, \quad 0 < \epsilon \ll 1,$$

(4.4)

and both $\alpha$ and $\Phi$ are smooth functions. Then equation (4.3) can be written in the form

$$U = \frac{1}{2\pi \epsilon} \int_{-\infty}^{\infty} \alpha(X) e^{i\phi} \, dX \quad \text{where } \phi = \frac{\Phi(X) - kX}{\epsilon}.$$

(4.5)

The tasks in this problem are as follows:

1. Use the method of stationary phase to obtain a leading order approximation for $U$ in (4.5) in the small $\epsilon$ limit. You should be able to show that

$$U \sim \frac{1}{\sqrt{\epsilon}} \beta(k) e^{i\phi(k)/\epsilon},$$

(4.6)

for some functions $\beta$ and $g$. Then (4.2) yields

$$u \sim \frac{1}{\sqrt{\epsilon}} \int_{-\infty}^{\infty} \beta(k) e^{i\psi} \, dk, \quad \text{where } \psi = \frac{kX - \omega T + g(k)}{\epsilon}, \quad T = \epsilon t, \quad \text{and } \omega = \Omega(k).$$

(4.7)

**Note:** Consider only the case where there is exactly one isolated root for the stationary phase equation — i.e.: in remark 4.1, the case where there is only one $x_s$, and $h''(x_s) \neq 0$.

2. Use the method of stationary phase to obtain a leading order approximation for $u$ in (4.7) in the small $\epsilon$ limit — this for “arbitrary” functions $\beta$ and $g$. You should be able to show that

$$u \sim A(X,T) e^{i\theta}, \quad \text{where } \theta = \frac{\Theta(X,T)}{\epsilon},$$

(4.8)

for some functions $A$ and $\Theta$. Again, consider only the case where there is exactly one isolated root for the stationary phase equation.
3. Show that, if you define the local wave numbers and wave frequencies by \( k = \Theta_X \) and \( \omega = -\Theta_T \) in (4.8), then \( \omega = \Omega(k) \). Hence
\[
0 = k_T + \omega_X = k_T + c_g(k)k_X,
\]
where \( c_g(k) = \frac{d\Omega}{dk} \) (4.9) is the group speed. What equation does \( A \) satisfy? You should be able to find a pde for \( A \), whose coefficients depend on \( k \), but not on \( X \) or \( T \) directly.

**Hint:** Examine the behavior of \( A \) along the characteristic lines \( dX/dT = c_g(k) \) — note that \( k \) is constant along them. The formula that you obtain for \( dA/dT \) will involve explicit dependence on \( T \), which you can eliminate in favor of \( k_x \). Now use \( dA/dT = A_T + c_g A_X \) to get a pde.

4. For the particular functions \( \beta = \beta(k) \) and \( g = g(k) \) found in item 1, show that the initial data is recovered. That is: \( A(X, 0) = \alpha(X) \) and \( \Theta(X, 0) = \Phi(X) \).

**Remark 4.1** Recall that the method of stationary phase tells us that, for an integral of the form
\[
I = \int_{-\infty}^{\infty} f(x) e^{i h(x) \nu},
\]
the dominant contribution as \( \nu \to \infty \) is provided by the stationary points \( x_s \) — solutions to \( h'(x) = 0 \). If this set is not empty, and \( h''(x_s) \neq 0 \) for every stationary point, then
\[
I \sim \sum_s \sqrt{\frac{2\pi}{\nu |h''(x_s)|}} f(x_s) \exp \left( i h(x_s) \nu + i \sigma \frac{\pi}{4} \right), \quad \text{as} \quad \nu \to \infty,
\]
where \( \sigma = \text{sign}(h''(x_s)) \). Stationary points where \( h''(x_s) = 0 \) yield larger contributions, for example: a point such that \( 0 = h'(x_s) = h''(x_s) \) and \( h''(x_s) \neq 0 \) contributes with a term that vanishes like \( \nu^{-1/3} \) — instead of \( \nu^{-1/2} \), as above.

4.2 **Answer: Wave modulation equations using Fourier Transforms**

**1a.** From (4.5), using (4.11) we have that
\[
U \sim \sqrt{\frac{1}{2\pi \epsilon |\Phi''(X_s)|}} \alpha(X_s) \exp \left( i \sigma \frac{\pi}{4} + i \frac{\Phi(X_s) - k X_s}{\epsilon} \right),
\]
where \( X_s = X_s(k) \) is the function defined by \( k = \Phi'(X_s) = k_0(X_s) \), and \( \sigma = \text{sign}(\Phi''(X_s)) \).

Equation (4.12) has the form in (4.6), where
\[
\beta = \sqrt{\frac{1}{2\pi |\Phi''(X_s)|}} \alpha(X_s) \exp \left( i \sigma \frac{\pi}{4} \right) \quad \text{and} \quad g = \Phi(X_s) - k X_s
\]
are both functions of \( k \) via \( X_s = X_s(k) \). In particular, note that
\[
g'(k) = -X_s(k) \quad \text{and} \quad g''(k) = -\frac{1}{\Phi''(X_s(k))}.
\]

**2a.** From (4.7), using (4.11) we have that
\[
u \sim \sqrt{\frac{2\pi}{|g''(k_s) - \Omega''(k_s)|}} \beta(k_s) \exp \left( i \frac{\pi}{4} + i \frac{k_s X - \omega_s T + g(k_s)}{\epsilon} \right),
\]
where \( k_s = k_s(X, T) \) is the function defined by \( 0 = X - c_g(k_s) T + g'(k_s) \), \( \mu = \text{sign}(g''(k_s) - \Omega''(k_s) T) \), \( \omega_s = \Omega(k_s) \), and \( c_g = \text{group speed} \).

This has the form in (4.8), with

\[
A = \sqrt{\frac{2\pi}{|g'''(k_s) - \Omega'''(k_s) T|}} \beta(k_s) \exp \left( i \mu \frac{\pi}{4} \right) \quad \text{and} \quad \Theta = k_s X - \omega_s T + g(k_s),
\]

(4.16)

where \( k_s = k_s(X, T) \), as above.

3a. From the equation defining \( k_s \), it follows that

\[
k = \Theta_X = (X - c_g(k_s) T + g'(k_s)) (k_s)_X + k_s = k_s,
\]

\[
-\omega = \Theta_T = (X - c_g(k_s) T + g'(k_s)) (k_s)_T - \omega_s = -\omega_s.
\]

Namely \( k = k_s \) and \( \omega = \omega_s = \Omega(k_s) = \Omega(k) \). Hence (4.9) applies.

From (4.9) it follows that \( k = k_s \) is constant along the characteristic lines defined by \( \frac{dX}{dT} = c_g(k) \). Hence, from (4.16), along the characteristics

\[
\frac{dA}{dT} = -\frac{1}{2} \frac{\Omega''(k_s)}{\Omega'''(k_s) T - g'''(k_s)} A = -\frac{1}{2} (c_g(k_s))_X A,
\]

(4.17)

where we have used that \((k_s)_X = \frac{1}{\Omega'''(k_s) T - g'''(k_s)}\), as follows from \( X = c_g(k_s) T - g'(k_s) \).

Equation (4.17) is equivalent to the following p.d.e.

\[
A_T + c_g(k) A_x + \frac{1}{2} (c_g(k))_X A = 0 \quad \text{or} \quad (A^2)_T + (c_g(k) A^2)_X = 0.
\]

(4.18)

for the complex wave amplitude \( A \). Note that, in terms of the polar form \( A = a e^{i \varphi} \), this is the same as

\[
(a^2)_T + (c_g(k) a^2)_X = 0 \quad \text{and} \quad \varphi_T + c_g(k) \varphi_X = 0,
\]

(4.19)

where the first equation says that the wave energy flows at the group speed, while the second states that the complex wave amplitude phase propagates at the group speed — just as \( k \) and \( \omega \) do.

4a. From (4.14), the equation defining \( k_s \) at time \( T = 0 \) reduces to \( X_s(k_s(X, 0)) = X \). From the equation defining \( X_s \) it then follows that \( k_s(X, 0) = k_0(X) \). Furthermore, using this in the definition of \( g \) in (4.13), we see that \( g(k_s(X, 0)) = \Phi(X) - k_0(X) X \), hence

\[
\Theta(X, 0) = \Phi(X),
\]

(4.20)

using the definition of \( \Theta \) in (4.16). Similarly, we have:

1. \( \beta(k_s(X, 0)) = \frac{1}{\sqrt{2\pi |g'''(X)|}} \alpha(X) \exp \left( i \sigma \frac{\pi}{4} \right) \), where \( \sigma = \text{sign}(\Phi''(X)) \).

2. \( g''(k_s(X, 0)) = -1/\Phi''(X) \). In particular, at \( T = 0, \mu = -\sigma \).

It should then be obvious that

\[
A(X, 0) = \alpha(X).
\]

(4.21)

THE END.