1 Acoustic impedance of a gas filled pipe

1.1 Statement: Acoustic impedance of a gas filled pipe

Consider the propagation of sound waves in a cylindrical, gas filled pipe, of cross-sectional area $S$. Let the conditions in the pipe be such that in the absence of sound waves, when the gas is at rest, the pressure is $p = p_0$ and the density $^1\rho = \rho_0$. Let $c_0$ be the sound speed.

Then, in terms of $\rho = \rho_0 + \delta\rho$, the equations governing the propagation of sound waves (acoustics) in the pipe are

$$\delta p_t + p_0 u_x = 0 \quad \text{and} \quad \rho_0 u_t + c_0^2 \delta p_x = 0,$$

(1.1)

$^1\rho = \text{mass per unit volume}.$
where \( u \) is the small flow velocity associated with the sound waves and \( \delta \rho \) is the small density perturbation produced by the sound waves. The pressure is given by
\[
p = p_0 + \delta p = p_0 + c_0^2 \delta \rho. \tag{1.2}
\]
Assume now a semi-infinite pipe, \( x > 0 \), apply a sinusoidal force \( F = a e^{i \omega t} \) at \( x = 0 \), and compute the resulting acoustic wave moving away from the end. Use this to obtain the acoustic impedance of the pipe.

**Notes/remarks.**

1. Apply the force via a vibrating piston/diaphragm at the end. Note that what is prescribed is the force on the piston, not its position or velocity. Hence: what is the appropriate boundary condition that you should use at \( x = 0 \) to solve the equations?

2. For linear problems with sinusoidal solutions, the impedance is defined as the (complex) coefficient relating the applied force to the system response:
\[
\text{complex amplitude of force} = Z \times \text{complex amplitude of response}, \tag{1.3}
\]
where \( Z \) is the impedance. Note that \( Z \) is generally a function of the applied wave-frequency \( \omega \). The impedance is important, in particular, to determine conditions at junctions that avoid reflections (impedance matching).

3. Note that equation (1.3) is a little ambiguous. When the system response is vector valued, how do we measure the “amplitude” of the response? For example, in our system here, should we look at the density response? Or maybe the velocity response? Something else?

The key to the answer is: for \( Z \) to be useful in determining no reflection conditions, it ought to be defined in terms of responses that matter for this purpose. Thus consider a diaphragm in the pipe, with one gas on one side and some other gas in the other. In this case the gases apply a force on each other via the pressure, with must match on each side. If the corresponding flow velocities then match, a wave can go through the diaphragm without reflections. Hence
\[
\text{For the case of a gas in a pipe, the appropriate variable to use on the right hand side of (1.3) is the flow velocity.} \tag{1.4}
\]
What if the pipe cross-section changes? If the cross-section change is abrupt, the hypotheses leading to 1-D acoustics break-down. On the other hand, if the change is slow, the concept of impedance as described here is not powerful enough: a characterization via a complex scalar, as in (1.3), makes sense for something that is very localized in space — not a slow transition over an extended region in space.\(^2\) This exercise does not deal with situations like this.

### 1.2 Answer: Acoustic impedance of a gas filled pipe

The appropriate boundary condition at \( x \) is
\[
F = \delta p S = c_0^2 S \delta \rho. \tag{1.5}
\]
Thus the desired solution is
\[
\delta \rho = \frac{p_0}{c_0} A e^{i \omega \left( t - \frac{x}{c_0} \right)}, \quad u = A e^{i \omega \left( t - \frac{x}{c_0} \right)}, \quad \text{and} \quad \delta p = \rho_0 c_0 A e^{i \omega \left( t - \frac{x}{c_0} \right)}, \tag{1.6}
\]
where
\[
a = \rho_0 c_0 S A. \quad \text{Hence} \quad Z = \rho_0 c_0 S. \tag{1.7}
\]

\(^2\) In particular, note that there cannot be any diaphragm/membrane in this case. This is a very different situation!
2 Group speed & average energy density flux #02

2.1 Statement: Group speed & average energy density flux #02

Consider a linear, homogeneous, dispersive system, with energy balance equation

\[ \mathcal{E}_t + \mathcal{F}_x = 0, \]  

(2.1)

where \( \mathcal{E} \) is the energy density and \( \mathcal{F} \) is the energy flux. Let now \( \mathcal{E}_{av} \) and \( \mathcal{F}_{av} \) be the average energy density and flux associated with a sinusoidal steady plane wave solution. Then, for systems of this type it is generally true that

\[ \mathcal{F}_{av} = c_g \mathcal{E}_{av}, \quad \text{where } c_g \text{ is the group speed.} \]  

(2.2)

For example, consider the equation

\[ u_{tt} - c^2 u_{xx} + \mu u_{xxxx} = 0, \]  

(2.3)

where \( c > 0 \) and \( \mu > 0 \) are constants. Then \( \omega^2 = m^2 + c^2 k^2 \),

\[ \mathcal{E} = \frac{1}{2} (u_t^2 + c^2 u_x^2 + \mu u_{xx}^2), \quad \mathcal{F} = -c^2 u_x u_t, \quad \text{and } c_g = c^2 \frac{k}{\omega}. \]  

(2.4)

Plugging into \( \mathcal{E} \) and \( \mathcal{F} \) a solution of the form \( u = a \cos(kx - \omega t + \theta_0) \), and computing the averages over one wave-period (or one wave-length) yields

\[ \mathcal{E}_{av} = \frac{1}{4} (\omega^2 + c^2 k^2 + m^2) a^2 = \frac{1}{2} \omega^2 a^2 \quad \text{and} \quad \mathcal{F}_{av} = \frac{1}{2} c^2 k \omega a^2. \]  

(2.5)

It should then be obvious that (2.2) applies.

For each of the examples below, verify that (2.2) holds.

1. Thin beam equation: \( u_{tt} + \mu u_{xxxx} = 0 \), where \( \mu > 0 \) is a constant. In this case the energy density is \( \mathcal{E} = \frac{1}{2} (u_t^2 + \mu u_{xx}^2) \), what is the energy flux \( \mathcal{F} \)?

2. Thin beam under tension: \( u_{tt} - c^2 u_{xx} + \mu u_{xxxx} = 0 \), where \( c, \mu > 0 \) are constants. In this case the energy density is \( \mathcal{E} = \frac{1}{2} (u_t^2 + c^2 u_x^2 + \mu u_{xx}^2) \), what is the energy flux \( \mathcal{F} \)?

3. Linear Boussinesq equation: \( \eta_{tt} - c^2 \eta_{xx} - \mu \eta_{xxtt} = 0 \), where \( c \) and \( \mu > 0 \) are constants. In this case \( \mathcal{E} = \frac{1}{2} \eta_t^2 + \frac{1}{2} c^2 \eta_x^2 + \frac{1}{2} \mu \eta_{xx}^2 \), what is \( \mathcal{F} \)?

2.1.1 Background for the equations in this exercise

1. Thin beam equation: \( u_{tt} + \mu u_{xxxx} = 0 \), where \( \mu > 0 \) is a constant.

This equation describes the small in-plane deviations from equilibrium of a straight, homogeneous, thin, elastic beam of uniform cross-section. The \( x \)-coordinate is placed along the beam’s axis, with \( y = u(x, t) \) parameterizing the deviations of the beam’s axis from straight. Further \( \mu = EI/m \), where \( E \) is Young’s elastic modulus, \( I \) is the second moment of area\(^5\) of the beam cross-section, and \( m \) is the mass per unit length of the beam.

2. Thin beam under tension: \( u_{tt} - c^2 u_{xx} + \mu u_{xxxx} = 0 \), where \( c, \mu > 0 \) are constants.

This equation applies under the conditions in example 1, if a tension \( T \) is applied along the beam axis. Then \( c = \sqrt{T/m} \).

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\(^{3}\) Linear constant coefficient equations with solutions proportional to \( e^{i(k x - \omega t)} \), where \( \omega = \omega(k) \) is real for \( k \) real.

\(^{4}\) Mechanical example: string under tension on an elastic bed.

\(^{5}\) \( I = \int_A y^2 dy dz \) where \( A \) is the beam cross-section and the \( x \)-axis is placed so that \( \int_A y dy dz = 0 \).
3. **Linear Boussinesq equation:** \( \eta_{tt} - c^2 \eta_{xx} - \mu \eta_{xxtt} = 0 \), where \( c, \mu > 0 \) are constants. This equation describes the behavior of long, linear, gravity waves in a rectangular channel of depth \( h_e \). Then \( c^2 = g h_e \) (\( g \) is the acceleration of gravity), \( \mu = (1/3) h_e^2 \), the total fluid depth is \( h = h_e + \eta \), and the horizontal flow velocity follows from

\[
  u_x = -\frac{1}{h_e} \eta_t \text{ and } u_t = -g \eta_x - \frac{1}{3} h_e \eta_{xxtt}.
\]

### 2.2 Answer: Group speed & average energy density flux #02

The items below correspond to the items in the problem statement.

1. **Thin beam equation:** \( u_{tt} + \mu u_{xxxx} = 0 \), where \( \mu > 0 \) is a constant. Multiplying the equation by \( u_t \) it is easy to arrive at

\[
  \frac{\partial}{\partial t} \left( \frac{1}{2} u_t^2 + \frac{1}{2} \mu u_{xx}^2 \right) + \frac{\partial}{\partial x} \left( \mu u_{xxx} u_t - \mu u_{xx} u_{tx} \right) = 0,
\]

from which \( E \) and \( F \) follow. Then, for a sinusoidal solution \( u = a \cos(k x - \omega t + \theta_0) \), where \( \omega^2 = \mu k^4 \) and \( c_g = 2 \omega/k \), we have

\[
  E_{av} = \frac{1}{4} \left( \omega^2 + \mu k^4 \right) a^2 = \frac{1}{2} \omega^2 a^2 \quad \text{and} \quad F_{av} = \mu k^3 \omega a^2 = \frac{\omega^3}{k} a^2 = c_g E_{av}.
\]

That is: (2.2) holds.

2. **Thin beam under tension:** \( u_{tt} - c^2 u_{xx} + \mu u_{xxxx} = 0 \), where \( c, \mu > 0 \) are constants. Multiplying the equation by \( u_t \) it is easy to arrive at

\[
  \frac{\partial}{\partial t} \left( \frac{1}{2} u_t^2 + \frac{1}{2} c^2 u_x^2 + \frac{1}{2} \mu u_{xx}^2 \right) + \frac{\partial}{\partial x} \left( \mu u_{xxx} u_t - \mu u_{xx} u_{tx} - c^2 u_x u_t \right) = 0,
\]

from which \( E \) and \( F \) follow. Then, for a sinusoidal solution \( u = a \cos(k x - \omega t + \theta_0) \), with \( \omega^2 = c^2 k^2 + \mu k^4 \) and \( c_g = \frac{1}{\omega}(c^2 k + 2 \mu k^3) \), we have

\[
  E_{av} = \frac{1}{4} \left( \omega^2 + c^2 k^2 + \mu k^4 \right) a^2 = \frac{1}{2} \omega^2 a^2 \quad \text{and} \quad F_{av} = \mu k^3 \omega a^2 + \frac{1}{2} c^2 k \omega a^2 = c_g E_{av}.
\]

That is: (2.2) holds.

3. **Linear Boussinesq equation:** \( \eta_{tt} - c^2 \eta_{xx} - \mu \eta_{xxtt} = 0 \), where \( c, \mu > 0 \) are constants.

Multiplying the equation by \( \eta_t \) it is easy to arrive at

\[
  E_t + F_x = 0,
\]

where

\[
  E = \frac{1}{2} \eta_t^2 + \frac{1}{2} c^2 \eta_x^2 + \frac{1}{2} \mu \eta_{xtt}^2 \quad \text{and} \quad F = -c^2 \eta_t \eta_x - \mu \eta_t \eta_{xxtt}.
\]

Then, for a sinusoidal solution \( u = a \cos(k x - \omega t + \theta_0) \), with \( \omega^2 = \frac{c^2 k^2}{1 + \mu k^2} \) and \( c_g = \frac{\omega}{k (1 + \mu k^2)} \), we have

\[
  E_{av} = \frac{1}{4} \left( \omega^2 + c^2 k^2 + \mu k^2 \omega^2 \right) a^2 = \frac{1}{2} c^2 k^2 a^2 \quad \text{and} \quad F_{av} = \frac{1}{4} \left( c^2 k \omega + \mu k \omega^3 \right) a^2 = c_g E_{av},
\]

where the last equality follows upon writing \( \omega^3 = \omega \omega^2 \), and using the formula giving \( \omega^2 \) in terms of \( k \).

That is: (2.2) holds.
3 Junction in a gas filled pipe

3.1 Statement: Junction in a gas filled pipe

Consider the propagation of sound waves in a cylindrical gas filled pipe such that

a. At \( x = 0 \) there is a thin, flexible and “mass-less”, diaphragm separating the gas in \( x < 0 \) from that in \( x > 0 \) — the two gases not being the same.\(^6\) We will call the gas on the left “gas 1”, and the gas on the right “gas 2”.

b. In the absence of sound waves the two gases are at rest, with common pressure \( p_0 \), and densities\(^7\) \( \rho_j \) (\( j = 1 \) or \( 2 \)). Let \( c_j \) be the corresponding sound speeds.

Then, in terms of \( \rho = \rho_j + \delta \rho \), the equations governing the propagation of sound waves (acoustics) in the pipe are

\[
\delta \rho_t + \rho_j u_x = 0 \quad \text{and} \quad \rho_j u_t + c_j^2 \delta \rho_x = 0,
\]

where: (i) \( j = 1 \) for \( x < 0 \) and \( j = 2 \) for \( x > 0 \); (ii) \( u \) is the small flow velocity associated with the sound waves; and (iii) \( \delta \rho \) is the small density perturbation associated with the sound waves. The pressure is given by

\[
p = p_0 + \delta p = p_0 + c_j^2 \delta \rho.
\]

Questions and tasks:

1. What conditions should be imposed at \( x = 0 \) to complete the equations?
2. Calculate the reflected and transmitted waves, due to the junction at \( x = 0 \), produced by an incident wave from the left. That is, consider a solution such that

\[
\delta \rho = \begin{cases} 
I(t - \frac{x}{c_1}) + R(t + \frac{x}{c_1}) & \text{for} \quad x < 0, \\
T(t - \frac{x}{c_2}) & \text{for} \quad x > 0,
\end{cases}
\]

where \( I = I(\xi) \neq 0 \) is some given periodic function with zero mean. Then

(i) Write the corresponding expression for \( u \). The fact that the diaphragm can vibrate, but not move away from \( x = 0 \), determines a constant showing up in the expression for \( u \).

(ii) Determine \( R \) and \( T \) in terms of \( I \).

3. When is \( R \equiv 0 \) (impedance matching)?
4. Is \( T \equiv 0 \) (total reflection) possible?

3.2 Answer: Junction in a gas filled pipe

The pressure and the fluid velocity must be continuous at \( x = 0 \), this because (i) A “mass-less” diaphragm cannot have a net force applied to it; and (ii) The fluid velocity on each side is equal to the diaphragm velocity. Hence

\[
[u] = 0 \quad \text{and} \quad [c_j^2 \delta \rho] = 0 \quad \text{at} \quad x = 0,
\]

where we use the brackets to indicate the jump in the enclosed quantity.

Given \( \delta \rho \) as in (3.3), to satisfy (3.1) it must be that

\[
u = \begin{cases} 
\frac{c_1}{p_1} I \left(t - \frac{x}{c_1}\right) - \frac{c_1}{p_1} R \left(t + \frac{x}{c_1}\right) & \text{for} \quad x < 0, \\
\frac{c_2}{p_2} T \left(t - \frac{x}{c_2}\right) & \text{for} \quad x > 0.
\end{cases}
\]

\(^6\) The difference could be as simple as same gas, but different temperature.

\(^7\) \( \rho = \text{mass per unit volume.} \)
In principle, we could add an arbitrary constant to \( u \), since (3.1) only determines \( u_t \) and \( u_x \) once \( \delta \rho \) is known. We set this constant to zero, and justify it later — see remark 3.1.

Substituting (3.3) and (3.5) into (3.4) yields

\[
R(\xi) = \frac{c_2 \rho_2 - c_1 \rho_1}{c_2 \rho_2 + c_1 \rho_1} I(\xi) \quad \text{and} \quad T(\xi) = \frac{c_1}{c_2} \frac{2 c_1 \rho_2}{c_2 \rho_2 + c_1 \rho_1} I(\xi). \tag{3.6}
\]

It follows that \( R \equiv 0 \) if and only if \( c_1 \rho_1 = c_2 \rho_2 \). Furthermore \( T = 0 \) cannot happen.

Remark 3.1 Because the diaphragm can vibrate, but not move away from \( x = 0 \), the average flow velocity at \( x = 0 \) should be zero. Thus the constant we could have added to \( u \) in (3.5) must vanish.

4 Junction in a string on an elastic bed

4.1 Statement: Junction in a string on an elastic bed

Consider the propagation of waves in an infinite string on an elastic bed, whose properties change across a junction at \( x = 0 \). Specifically we assume that both the mass per unit length \( \rho \) — as well as the bed elastic constant \( b \), are piece-wise constant

\[
\rho = \rho_1 \quad \text{and} \quad b = b_1 \quad \text{for} \quad x < 0, \quad \text{and} \quad \rho = \rho_2 \quad \text{and} \quad b = b_2 \quad \text{for} \quad x > 0, \tag{4.1}
\]

where the \( \rho_j \) and the \( b_j \) are all positive constants. The spring tension \( T \) is, of course, uniform throughout — else longitudinal motion along the string would have to be considered as well. The equations are then

\[
\begin{align*}
  u_{tt} - c_1^2 u_{xx} &= -\omega_1^2 u, \quad \text{for} \quad x < 0, \tag{4.2} \\
  u_{tt} - c_2^2 u_{xx} &= -\omega_2^2 u, \quad \text{for} \quad x > 0, \tag{4.3}
\end{align*}
\]

where \( c_j = \sqrt{T/\rho_j} \), and \( \omega_j = \sqrt{b_j/\rho_j} \). These equations are supplemented by junction conditions, where we enforce continuity of the string, and of the transversal forces (given by \( T u_x \)). Thus

\[
u \quad \text{and} \quad u_x \quad \text{are continuous at} \quad x = 0. \tag{4.4}
\]

Consider now the situation where, somewhere far to the left, and incoming monochromatic wave, of unit amplitude, is being generated. Namely

\[
u = \exp (ik x - i \omega t), \tag{4.5}
\]

where \( k > 0 \) is a constant, and \( \omega = \sqrt{c_1^2 k^2 + \omega_1^2} \). YOUR TASKS ARE NOW:

1. Assume that \( \omega > \omega_2 \). Then calculate the reflected wave \( R(k) \exp (-ik x - i \omega t) \) and the transmitted wave \( T_c(k) \exp (i K x - i \omega t) \), where \( K = K(k) > 0 \). This so that

\[
u = \begin{cases} 
\exp (ik x - i \omega t) + R(k) \exp (-ik x - i \omega t) & \text{for} \quad x < 0 \\
T_c(k) \exp (i K x - i \omega t) & \text{for} \quad x > 0
\end{cases} \tag{4.6}
\]

solves (4.2 – 4.4)
2. For each of the waves in item 1, calculate the average energy flux $F = \langle -T u_t u_x \rangle$, and check that the energy budget works as it should.

**IMPORTANT:** When computing energy fluxes, remember that, in fact, it is the real part of (4.5 – 4.6) that is the object of interest!

3. Assume that $\omega < \omega_2$. Then compute the reflected wave on $x < 0$, and the evanescent wave on $x > 0$. Again, check the energy budget.

4. Are there any conditions under which there is no reflection (impedance matching)?

### 4.2 Answer: Junction in a string on an elastic bed

1. **Case $\omega > \omega_2 > 0$.** The condition on $K$ for (4.6) to be a solution is $\omega^2 = c_2^2 K^2 + \omega_2^2$ — dispersion relation. This has the positive solution

   $$K = \frac{1}{c_2} \sqrt{\omega^2 - \omega_2^2} = \frac{1}{c_2} \sqrt{c_2^2 K^2 + \omega_2^2 - \omega_2^2} > 0.$$  

   (4.7)

   In addition, the conditions in (4.4) must apply. Hence

   $$1 + R = \mathcal{T}_c \quad \text{(continuity of $u$)} \quad \text{and} \quad k - k R = K \mathcal{T}_c \quad \text{(continuity of $u_x$).}$$  

   (4.8)

   These equations then yield (note that both $R$ and $\mathcal{T}_c$ are real valued)

   $$R = \frac{k - K}{k + K} \quad \text{and} \quad \mathcal{T}_c = \frac{2k}{k + K}. \quad (4.9)$$

2. For $x > 0$, (4.6) yields (upon taking real parts)

   $$u = 2 \mathcal{T}_c \cos(K x - \omega t) = 2 \mathcal{T}_c \cos \theta_r, \quad -T u_t u_x = 4 \omega K T \mathcal{T}_c^2 \sin^2 \theta_r,$$

   so that the average energy flux on the right is

   $$F_r = 2 \omega K T \mathcal{T}_c^2. \quad (4.10)$$

   Similarly, for $x < 0$, taking real parts yields

   $$u = 2 \cos \theta^+_l + 2 R \cos \theta^+_r, \quad \text{where} \quad \theta^+_l = k x \pm \omega t.$$  

   From this it follows that

   $$u_t = 2 \omega (\sin \theta^+_l - R \sin \theta^+_r) \quad \text{and} \quad u_x = -2k (\sin \theta^+_l + R \sin \theta^+_r).$$

   Hence

   $$-T u_t u_x = 4 \omega k T (\sin^2 \theta^+_l - R^2 \sin^2 \theta^+_r),$$

   so that the average energy flux on the left is

   $$F_\ell = 2 \omega k T \left( 1 - R^2 \right). \quad (4.11)$$

   **Note that the average energy flux on the left is the sum of the individual fluxes each wave would have if alone.** Because fluxes are quadratic in the solution, it would seem that this should not be true in general. However, this is a generic result. The reason is that the average of products of sinusoidal functions with different phases vanishes.

   As a simple example illustrating this, compute the average of $u^2$, where $u$ is the real part of $a e^{i \theta_1} + b e^{i \theta_2}$.

   It is easy to see that the cross-product terms contribute only if either $\theta_1 + \theta_2 = 0$ or $\theta_1 - \theta_2 = 0$.

   **Conservation of energy requires $F_\ell = F_r$.** From (4.10 – 4.11) this is equivalent to

   $$k R^2 + K \mathcal{T}_c^2 = k, \quad (4.12)$$

   which is satisfied by (4.9).

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\(^a\) Average over one time period.
3. Case $\omega_2 > \omega > 0$. Since then $\omega_2^2 > \omega^2 = \omega_1^2 + c_1^2 k^2$, this case requires

$$\omega_2 > \omega_1, \quad \text{with} \quad 0 < k < \frac{1}{c_1} \sqrt{\omega_2^2 - \omega_1^2}. \quad (4.13)$$

The solution then has an evanescent wave for $x > 0$, and has the form

$$u = \begin{cases} 
\exp (i k x - i \omega t) + R(k) \exp (-i k x - i \omega t) & \text{for} \quad x < 0 \\
T_c(k) \exp (-\lambda x - i \omega t) & \text{for} \quad x > 0
\end{cases} \quad (4.14)$$

where $\lambda = \lambda(k) > 0$ is defined by the “dispersion” relation on $x > 0$. That is $\omega^2 = \omega_2^2 - c_2^2 \lambda^2$, which yields

$$\lambda = \frac{1}{c_2} \sqrt{\omega_2^2 - \omega^2} = \omega_2^2 - \omega_1^2 - c_1 k^2. \quad (4.15)$$

The conditions in (4.4) then become

$$1 + R = T_c (\text{continuity of } u) \quad \text{and} \quad i (k - k R) = -\lambda T_c (\text{continuity of } u_x). \quad (4.16)$$

Thus

$$R = \frac{k - i \lambda}{k + i \lambda} \quad \text{and} \quad T_c = \frac{2 k}{k + i \lambda}. \quad (4.17)$$

The average energy fluxes on the right and left are now

$$F_r = 0 \quad \text{and} \quad F_\ell = 2 \omega k T \left(1 - |R|^2\right) = 0, \quad (4.18)$$

where we have used that $R$ is a complex number in the unit circle. As expected, conservation of energy holds.

4. Impedance matching (no reflection) occurs for the case in item 1 when $k = K$. Using (4.7), we see that this is equivalent to

$$c_2^2 k^2 = c_1^2 k^2 + \omega_1^2 - \omega_2^2 \iff k = \sqrt{\frac{\omega_1^2 - \omega_2^2}{c_1^2 - c_2^2}}. \quad (4.19)$$

This, of course, can only happen for $(\omega_1 - \omega_2) (c_1 - c_2) < 0$, meaning that: the stiffer string (larger $c_j$) has to be on a softer bed (lower $\omega_j$).

Of course, the case in item 3 is the exact opposite of impedance matching. In this vein, notice that (4.9) does not give total reflection for any finite ($k > 0$) wave-length.

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5 Two strings connected by a third one

5.1 Statement: Two strings connected by a third one

Consider two semi-infinite strings under tension, connected by a short piece of another string. If $u = u(x, t)$ denotes the transverse deviations of the strings, in the linear (small deviations and small slopes), the equation describing the system is

$$0 = u_{tt} - c^2 u_{xx} \quad (5.1)$$
where \( c = c_1 \) for \( x < 0 \), \( c = c_2 \) for \( 0 < x < L \), \( c = c_3 \) for \( L < x \), the \( c_j > 0 \) are constants, and \( L \) is the length of the connecting piece. This equation must be supplemented by the conditions

\[
 u \quad \text{and} \quad u_x \quad \text{are continuous across the joints at} \quad x = 0 \quad \text{and} \quad x = L. 
\]  

(5.2)

Assume now that a monochromatic wave of wave-frequency \( \omega \) arrives at the (composite) junction. Then:

1. **Calculate the reflection and transmission coefficients across the (composite) junction.** That is, find solutions to the equation of the following form:

\[
 u = \begin{cases} 
 e^{i \omega (t-x/c_1)} + Re^{i \omega (t+x/c_1)} & \text{for} \ x < 0 \\
 a e^{i \omega (t-x/c_2)} + be^{i \omega (t+x/c_2)} & \text{for} \ 0 < x < L \\
 T_e e^{i \omega (t-x/c_3)} & \text{for} \ L < x 
\end{cases} 
\]  

(5.3)

where \( \omega > 0 \), \( a \), \( b \), \( R \), and \( T_e \) are constants. Note: \( \omega \) is a free constant, everything else is a function of \( \omega \).

**Important:** Make sure that your calculations do not involve division by zero!

2. **Why do all the terms in (5.3) have the same time dependence?** How does this follow from (5.2)?

3. **Under which conditions is there no reflection, \( R = 0 \) (i.e.: impedance matching)?**

4. **Is it possible to get no transmission, \( T_e = 0 \)?**

This problem illustrates the principle behind lens coating, used in optics to prevent reflection losses in complex multi-lense devises, — in a very simplified context.

### 5.2 Answer: Two strings connected by a third one

#### Part 2

The equations in (5.2) must hold for all times, at \( x = 0 \) and at \( x = L \). Hence, when the waves have an exponential dependence in time (as in this problem), (5.2) can apply only if all the exponentials are the same. Hence the use of the same time dependence for all the terms in (5.3).

Physically: if the piece of string immediately to the left of some point \( x_\ast \) vibrates with frequency \( \omega \), the piece immediately to the right of \( x_\ast \) (which is attached to the piece on the left) must vibrate with the same frequency. Hence, the whole string must vibrate with the same frequency. These arguments apply to a string in "steady" state only, else we cannot even talk about "frequency".

#### Part 1

Substituting (5.3) into (5.2) yields the following set of equations

\[
 1 + R = a + b \quad \text{and} \quad 1 - R = \frac{c_1}{c_2} (a - b) 
\]  

(5.4)

from the \( x = 0 \) equation, and

\[
 A + B = D \quad \text{and} \quad A - B = \frac{c_2}{c_3} D \quad \iff \quad A = \frac{c_3 + c_2}{2c_3} D \quad \text{and} \quad B = \frac{c_3 - c_2}{2c_3} D 
\]  

(5.5)

from the \( x = L \) equation, where

\[
 A = a e^{-i \phi}, \quad B = a e^{i \phi}, \quad D = T_e e^{-i \psi}, \quad \phi = \omega L/c_2, \quad \text{and} \quad \psi = \omega L/c_3. 
\]  

(5.6)

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9 Given by \( c_j = \sqrt{T/\rho_j} \), where \( T \) is the (common) string tension and \( \rho_j \) are the respective string densities. As usual, we assume a tension that is constant throughout, so that neglecting longitudinal movements is justified.
Hence
\[ a = \frac{c_3 + c_2}{2c_3} D e^{i\phi} \quad \text{and} \quad b = \frac{c_3 - c_2}{2c_3} D e^{-i\phi}. \] (5.7)

But (5.4) is equivalent to
\[ 1 = \frac{c_2 + c_1}{2c_2} a + \frac{c_2 - c_1}{2c_2} b \quad \text{and} \quad R = \frac{c_2 - c_1}{2c_2} a + \frac{c_2 + c_1}{2c_2} b. \] (5.8)

The first of these two equations then yields
\[ e^{i\psi} = \left( \frac{c_2 + c_1}{2c_2} \frac{c_3 + c_2}{2c_3} e^{i\phi} + \frac{c_2 - c_1}{2c_2} \frac{c_3 - c_2}{2c_3} e^{-i\phi} \right) T_c, \] (5.9)

which determines the transmission coefficient \( T_c \). Then, from the second equation
\[ R = \left( \frac{c_2 - c_1}{2c_2} \frac{c_3 + c_2}{2c_3} e^{i\phi} + \frac{c_2 + c_1}{2c_2} \frac{c_3 - c_2}{2c_3} e^{-i\phi} \right) T_c e^{-i\psi}, \] (5.10)

which gives the reflection coefficient \( R \).

**Remark 5.1** Note that \((c_2 + c_1) (c_3 + c_2) > |(c_2 - c_1) (c_3 - c_2)|\), since all the \( c_j \)'s are positive. Thus, the coefficient in front of \( T_c \) on the right in (5.9) never vanishes.

Hence the transmission coefficient is always defined. In fact \( 0 < |T_c| < \infty \).

Alternatively, introduce
\[ R_{1j} = \frac{c_i - c_j}{c_i + c_j} \quad \text{and} \quad \gamma = \frac{4c_2 c_3}{(c_1 + c_2)(c_2 + c_3)} = (1 - R_{12}) (1 - R_{23}), \] (5.11)

where we notice that \(-1 < R_{1j} < 1\). Then the equations above take the form
\[ e^{i\psi} = \frac{e^{i\phi}}{\gamma} \left( 1 + R_{12} R_{23} e^{-i2\phi} \right) T_c \quad \text{and} \quad R = -\frac{e^{i\phi}}{\gamma} \left( R_{12} + R_{23} e^{-i2\phi} \right) T_c e^{-i\psi}. \] (5.12)

Hence
\[ T_c = \frac{\gamma e^{i(\psi - \phi)}}{1 + R_{12}R_{23} e^{-i2\phi}} \quad \text{and} \quad R = -\frac{R_{12} + R_{23} e^{-i2\phi}}{1 + R_{12}R_{23} e^{-i2\phi}}, \] (5.13)

where we note that the coefficients depend on \( \omega L \) via \( \psi \) and \( \phi \).

**Part 3**

**No reflection, \( R = 0 \).** This can happen only for \( R_{12} = \pm R_{23} \), which leads to the following cases

**Case** \( c_3 = c_1 \). Then \( R_{12} = -R_{23} \).

The **no-reflection condition** is \( \phi = \frac{\omega L}{c_z} = n \pi \), where \( n \) is an integer. That is:

A wave in the connecting segment travels back and forth (distance \( 2L \)) in \( n \) wave periods.

Not an interesting case, since no connection segment is needed at all to suppress reflection!

**Case** \( c_2^2 = c_1 c_3 \). Then \( R_{12} = R_{23} \).

The **no-reflection condition** is \( \phi = \frac{\omega L}{c_2} = \left( n + \frac{1}{2} \right) \pi \), where \( n \) is an integer. That is:

A wave in the connecting segment travels back and forth (distance \( 2L \)) in \( n + \frac{1}{2} \) wave periods. This situation corresponds to the “simplest” explanation of how come there is no reflection. The idea is that, because of the 1/2 period fraction in the travel time, the waves reflected from the first interface (at \( x = 0 \)), and those from the second (at \( x = L \)), have a \( \pi \) phase difference, and hence cancel each other. This is, basically, correct, but it is also an over-simplification. It predicts no reflection for any value of \( c_2 \), which (as shown here) is not true.
Part 4

From remark 5.1, it follows that **there is always transmission. Total reflection is not possible.**

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THE END.