1 Edge waves for a piece-wise constant wave equation

Statement: Edge waves for a piece-wise constant wave equation

Consider the wave equation

\[ u_{tt} - c^2 (u_{xx} + u_{yy}) = 0, \quad \text{for } y > 0, \tag{1.1} \]

where \( c = c_2 \) for \( y > L \), \( c = c_1 \) for \( 0 < y < L, c_2 > c_1 > 0 \) are constants, and \( L > 0 \) is a constant. Furthermore, \( u \) and \( u_y \) are continuous across the \( y = L \) interface, and the boundary condition \( u = 0 \) applies at \( y = 0 \).

Find all the edge waves for this problem. Namely, non-vanishing solutions to the problem above of the form

\[ u = U(y) \exp (i (k x - \omega t)), \tag{1.2} \]

where \( k > 0 \) and \( \omega \) are real constants, and \( U \) vanishes (exponentially) as \( y \to \infty \).

Hint. For any given fixed \( k \), the problem will lead to an eigenvalue problem, with eigenvalue \( \omega^2 \), and eigenfunction \( U = U(y) \). This problem can be then reduced to a transcendental equation that \( \omega^2 \) must satisfy. In order to study the solutions to this later equation, it may be useful to write it down in terms of the variable \( \Delta = (L/c_1) \sqrt{\omega^2 - k^2 c_2^2} \), which is restricted to the range

\[ 0 < \Delta < \Delta_M = k (L/c_1) \sqrt{c_2^2 - c_1^2}, \tag{1.3} \]
Reflected wave from an active boundary #01

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Calculate the reflected wave for the linear wave equation problem in figure 2.1. Note that:

1. There are no waves on \( x < 0 \). The equation applies for \( x > 0 \) only.
2. There is a critical angle \( \theta_c \) at which something very special happens.
   Find \( \theta_c \), and explain the physical meaning of what you found.
3. Show also that the model is stable: none of the normal modes grows — see remark 2.1.

Figure 2.1: Reflected wave from an active boundary. Here \( c, \gamma > 0 \) are constants. The incident wave has the form \( I = \exp (i k (-x \cos \theta + y \sin \theta - c t)) \), with \( 0 < \theta < \pi/2 \) and \( k \neq 0 \) constants.

The equation to be solved is

\[
\Phi_{tt} - c^2 \Delta \Phi = 0 \quad \text{for} \quad x > 0,
\]

with the boundary condition

\[
\Phi_{xt} + \gamma^2 \Phi_{yy} = 0 \quad \text{at} \quad x = 0,
\]

This is a made-up mathematical model for something that happens when there is an “active” boundary — i.e.: extra energy is available there. The motivating example here is combustion, where the boundary is a plane detonation wave (where a chemical reaction occurs) and we look at the linearized problem near this solution. The “real” problem is far more complicated than the model here, with more waves (here only the acoustic waves have been allowed to survive), and more variables. But the basic wave phenomena that occurs is the same that you will find here.

Remark 2.1 Because the equation and boundary condition are translational invariant in the \( y \)-direction, the problem can be Fourier Transformed in this variable, and we can write

\[
\Phi = \int_{-\infty}^{\infty} \phi(\ell, x, t) e^{i \ell y} d\ell,
\]

where \( \phi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi e^{-i \ell y} d\ell \)
satisfies
\[ \phi_{tt} - \ell^2 \phi_{xx} + \gamma^2 \ell^2 \phi = 0 \quad \text{for} \quad x > 0, \]
and
\[ \phi_{tt} - \gamma^2 \ell^2 \phi = 0 \quad \text{at} \quad x = 0, \]
where \(-\infty < \ell < \infty.\)

The normal modes are solutions of this problem of the form \( \phi = \varphi(\ell, x)e^{\lambda t}, \) decaying as \( x \to \infty, \) where \( \lambda \) is a constant. Since the equation for \( \varphi \) is a constant coefficients ode, it must be \( \varphi \propto e^{-\alpha x}, \) where \( \alpha \) is a constant such that \( \Re(\alpha) > 0. \)

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3 Conservation laws in elasticity

This problem is from the Continuum Media and Elasticity notes.

Statement: Conservation laws in elasticity

Note: despite the problem title, this exercise applies to any continuum media.

Consider a continuum media (solid or liquid) in some region of space, with an inertial cartesian coordinate system \( \vec{x} \). Let the media have the following fields associated with it: \( \rho = \rho(\vec{x}, t) = \) mass density, \( \vec{v} = \vec{v}(\vec{x}, t) = \) mass flow velocity, \( \tau = \tau(\vec{x}, t) = \) stress tensor, \( \vec{f} = \vec{f}(\vec{x}, t) = \) body forces per unit mass, and \( V = V(\vec{x}, t) = \) internal energy per unit volume. Assume that all these fields are smooth functions of \( (\vec{x}, t), \) and:

1. Derive pde that the fields above should satisfy. Use conservation of mass, (linear) momentum, and energy.

2. Use the equations in item 1 to obtain expressions for the flow velocity and internal energy material derivatives. That is, for:
   \[
   \frac{D\vec{v}}{Dt} = \vec{v}_t + (\vec{v} \cdot \nabla)\vec{v} \quad \text{and} \quad \frac{DV}{Dt} = V_t + (\vec{v} \cdot \nabla)V.
   \]
   Note that in the equation for \( \frac{DV}{Dt} \) the body forces should not appear.

3. Write the equation for the conservation of angular momentum and show that, given the equations in item 1, .................. it is satisfied if and only if \( \tau \) is symmetric.

Remark 3.1 The equations above have to be completed with appropriate equations of state. For example, in gas dynamics \( \tau \) is given in terms of the pressure \( p \) and \( \vec{v}, \) while \( V \) is given in terms of \( p \) and \( \rho. \) In elasticity \( \tau \) is a function of the strain tensor.

Recall that the stress tensor is defined as follows: for any surface with unit normal \( \hat{n}, \) the force per unit area (by the media on the side the normal points towards, onto the other side) is given by \( \tau \cdot \hat{n} = \{\tau_{ij} n_j\} \) — where we use the repeated index summation convention.

Important: for parts 1-2 be careful to not use the symmetry of \( \tau, \) since then part 3 would become pointless.

Notation: Let \( \vec{a} \) and \( \vec{b} \) be vectors, \( c \) a rank-two tensor, and \( \epsilon_{knm} \) the permutation multi-index. Then

1. \( \text{div}(\vec{a}) = (a_j)_{,j} = \) a scalar.
2. \( \text{div}(c) = \{c_{ij}\}_{,j} = \) a vector.
3. \( \vec{a} \otimes \vec{b} \) is a rank-two tensor defined by \( (\vec{a} \otimes \vec{b})_{ij} = a_i b_j. \)
4. \( \vec{a} \cdot c = \{a_i c_{ij}\} \) and \( c \cdot \vec{a} = \{c_{ij} a_j\} = \) both vectors. Generally, \( \vec{a} \cdot c \neq c \cdot \vec{a} \) (unless \( c \) is symmetric).
5. \( \vec{a} \times \vec{b} = \{\epsilon_{jnm} a_n b_m\} = \) a vector.
6. \( \vec{a} \times c \) and \( c \times \vec{a} \) are rank-two tensors defined by \( (\vec{a} \times c)_{ij} = \{\epsilon_{inm} a_n c_{mj}\}, \) and \( (c \times \vec{a})_{ij} = \{\epsilon_{inm} c_{jn} a_m\}. \)
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\varepsilon_{knm}$ is defined by: $\varepsilon_{knm} = 1$ if $(k, n, m) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$; $\varepsilon_{knm} = 0$ if two of the indexes are equal; $\varepsilon_{knm} = -1$ otherwise.

THE END.