18.376 Problem Set #04, MIT Spring 2019

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Due: Fri April 12, 2019.
Turn it in (by 3PM) at the Math. Problem Set Boxes, right outside room 4-174. There is a box/slot there for 376. Be careful to use the right box (there are many slots).

Note. Answers to all the problems will be posted, but only some problems will be graded.

The graded problems are “quiz #04”.

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1 Gravity water waves (dispersion relation)

Statement: Gravity water waves (dispersion relation)

The equations for (infinitesimal) irrotational surface waves on a liquid over a flat impermeable bottom, when surface tension and dissipative effects are neglected, are

\[ \Delta \Phi = 0, \text{ for } 0 < z < h \] (incompressibility). (1.1)
\[ \Phi_z = 0, \text{ for } z = 0 \] (impermeable bottom). (1.2)
\[ \eta_t - \Phi_z = 0, \text{ for } z = h \] (kinematic boundary condition). (1.3)
\[ \Phi_t + g \eta = 0, \text{ for } z = h \] (dynamic boundary condition). (1.4)

Here (i) \( \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2 \) is the Laplace operator, (ii) \( \Phi = \Phi(x, y, z, t) \) is the velocity potential — the flow velocity is given by \( \vec{u} = \text{grad} \Phi \), (iii) \( \vec{x} = (x, y) \) are the horizontal coordinates, (iv) \( z \) is the vertical coordinate — \( z = 0 \) is
the bottom and \( z = h \) is the equilibrium level for the liquid (\( h \) is a constant), (v) \( \eta \) is the deviation from equilibrium of the surface — the surface is at \( z = h + \eta(x, y, t) \), and (vi) \( g \) is the acceleration of gravity. Equation (1.2) is the statement that there is no flow through the bottom, equation (1.3) states that the velocity of the surface normal to itself is equal to the flow velocity normal to the surface, and equation (1.4) follows from the balance of forces at the interface — Bernoulli’s principle.

**Compute the dispersion relation for these equations:** separate the time and horizontal dependence as \( e^{i \theta} \) — where \( \theta = \vec{k} \cdot \vec{x} - \omega t \) and \( \vec{x} = (x, y) \), solve for the vertical dependence, and find the equation that relates \( \omega \) and \( \vec{k} \).

## 2 Narrow band wave packages #01

### 2.1 Statement: Narrow band wave packages #01

Consider a solution to a 1-D linear dispersive system with a narrow band spectrum and a single branch of the dispersion relation active.\(^1\) Using the Fourier Transform, this means that the solution can be written in terms of a scalar function of the form

\[
u = u(x, t) = \int_{-\infty}^{\infty} U(k) e^{i(kx - \omega t)} dk, \quad \text{where:} \tag{2.1}
\]

a. The wave-frequency \( \omega = \omega(k) \) is given by the dispersion relation. Assume that \( \omega \) is a smooth, real valued, function.

b. The complex amplitude \( U \) is concentrated near some wave-number \( k_0 \).

That is\(^2\)

\[
U = \frac{1}{\epsilon} A \left( \frac{k}{\epsilon} - k_0 \right), \tag{2.2}
\]

where \( 0 < \epsilon \ll 1 \) and \( A \) is a smooth function that decays rapidly at infinity.

*We use a-dimensional variables, otherwise a statement like \( 0 < \epsilon \ll 1 \) has no meaning.*

Assume that \( \frac{d^2 \omega}{dk^2}(k_0) = 0 \) and \( \mu_0 = \frac{d^3 \omega}{dk^3}(k_0) \neq 0 \). Thus the Taylor expansion for \( \omega \) centered at \( k_0 \) has the form

\[
\omega(k) = \omega_0 + c_0 (k - k_0) + \frac{1}{6} \mu_0 (k - k_0)^3 + \ldots, \tag{2.3}
\]

where \( \omega_0 = \omega(k_0) \) and \( c_0 = c_g(k_0) = \frac{d \omega}{dk}(k_0) \) is the group speed at \( k_0 \). Then show that \( u \) in (2.1) has the form of a modulated carrier wave

\[
u = a(X, T) e^{i(k_0 x - \omega_0 t)}, \tag{2.4}
\]

where \( X = \epsilon x \), \( T = \epsilon t \), and the modulation amplitude satisfies the equation

\[
a_T + c_0 a_X - \frac{1}{6} \epsilon^2 \mu_0 a_{XXX} = O(\epsilon^3). \tag{2.5}
\]

**What equation does \( a \) satisfy** in terms of the variables \( X = X - c_0 T \) and \( T = \epsilon^2 T = \epsilon^3 t \)?

*Hint. Write \( k = k_0 + \epsilon \kappa \) and substitute this into (2.2–2.3). Then use the result in (2.1).*

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\(^1\) If more than one exists.

\(^2\) The purpose of the pre-factor \( 1/\epsilon \) is so that the integral of \( U \) does not vanish as \( \epsilon \to 0 \).
3 String on an elastic bed: harmonic forcing at the critical frequency

Before doing this problem, check the Lecture topics for 18.376 notes: Section: Radiation damping. Subsection: Semi-infinite string over elastic bed with mass-spring at end. Subsubsection: Harmonic forcing.

Statement: String on an elastic bed: harmonic forcing at the critical frequency

Consider a semi-infinite string over an elastic bed, under tension, with a forced mass-spring system attached at its end (assume also small, in-plane, motion). With properly selected a-dimensional variables, the equations are

\[ u_{tt} - u_{xx} + u = 0 \quad \text{for } x > 0, \]
\[ u_{tt} + \Omega^2 u = 2\nu u_x + G \quad \text{at } x = 0, \]

where \(\Omega\) and \(\nu\) are positive constants, \(G = G(t)\) is the force applied to the mass attached to the string, and \(2\nu u_x(0, t)\) is the force by the string (due to its tension) on the mass. We will make the following assumptions:

a1. The forcing is harmonic, specifically: \(G = e^{i\omega t}\), with \(0 < \omega < 1\).

a2. The following applies:

\[ \Omega^2 < 1. \]

A particular solution to (3.1–3.3) is given by

\[ u_p = a e^{i\omega t - \ell x}, \quad \text{with } a = 1/(\Omega^2 + 2\nu\ell - \omega^2), \]

where \(\ell = \sqrt{1 - \omega^2}\).

Below we show how to use this solution to generate the solution to the initial value problem for (3.1–3.3). However, note: there is a critical value of \(\omega\), \(\omega_c\), at which (3.5) fails — the value such that \(\Omega^2 + 2\nu\ell - \omega^2 = 0\).

Remark 3.1 \(\Omega^2 + 2\nu\ell - \omega^2 = 0\) has exactly one solution for \(0 < \omega < 1\), \(\omega_c\).

Proof. Let \(f(\omega) = \omega^2 - 2\nu\ell\). This function is increasing, and satisfies \(f(0) = -2\nu < \Omega^2\) and \(f(1) = 1 > \Omega^2\).

Problem task: Find a particular solution for the case \(\omega = \omega_c\).

Hint. Use the technique illustrated by the following example: Consider the ode: \(\ddot{y} + y = e^{i\omega t}\) [A]. This has the solution \(y_p = (1 - \omega^2)^{-1}e^{i\omega t}\), valid as long as \(\omega^2 \neq 1\). To find a solution for \(\omega = 1\), notice that \(z = e^{i\omega t}\) satisfies, for any \(\omega\), the ode: \(\ddot{z} + z = (1 - \omega^2)e^{i\omega t}\) [B]. Now let \(\xi = \frac{\partial z}{\partial \omega}\) and take the derivative of [B] with respect to \(\omega\). This yields the equation: \(\dddot{\xi} + \xi = -2\omega e^{i\omega t} + it(1 - \omega^2)e^{i\omega t}\) [C]. Evaluating now [C] at \(\omega = 1\) leads to the desired particular solution, specifically: \(y_p = -\frac{1}{2}\xi(\omega = 1) = -\frac{1}{2}i\ell e^{it}\).

Note also that the answer to this problem is just slightly longer than this hint.

From particular to the general solution.

Here we show how to use a particular solution of (3.1–3.3), to reduce the initial value problem to one that can be solved by ode techniques and Fourier Transforms. Note: you do not need to read this to do the problem!

Disclaimer: the approach presented below is probably not “the best”. Think of it as a proof of concept only.

We begin by writing \(u = u_p + w\), where \(w\) solves (3.1–3.2) with \(G = 0\) and initial data:

\[ w(x, 0) = w_0(x) = u(x, 0) - u_p(x, 0) \quad \text{and} \quad w_t(x, 0) = w_1(x) = u_t(x, 0) - (u_p)_t(x, 0). \]

Introduce now \(v = v(x, t)\) by

\[ v = \mathcal{L} w = w_{xx} - (1 - \Omega^2)w - 2\nu w_x, \]

by the equation. Then

\[ v_{tt} - v_{xx} + v = 0 \quad \text{for } x > 0, \quad \text{with } v(0, t) = 0. \]

The initial conditions for this equation \(v(x, 0) = v_0(x)\) and \(v_t(x, 0) = v_1(x)\) follow from (3.6–3.7).

Why is it \(v(0, t) = 0\)? This is because \(w\) satisfies (3.1), so that \(v = w_{tt} + \Omega^2 w - 2\nu w_x\) as well.

Then

\[ v = \int_0^\infty \left( \tilde{v}_0(k) \sin(kx) \cos \left( \frac{\sqrt{1 + k^2} t}{\sqrt{1 + k^2}} \right) + \tilde{v}_1(k) \sin(kx) \frac{\sin \left( \frac{\sqrt{1 + k^2} t}{\sqrt{1 + k^2}} \right)}{\sqrt{1 + k^2}} \right) dk, \]

where \(\tilde{v}_0, \tilde{v}_1\) are determined by the initial conditions.
where \( \hat{v}_0 \) and \( \hat{v}_1 \) are the sine-Fourier Transforms of \( v_0 \) and \( v_1 \).

The issue is now: **Given \( v \), how do we recover \( w \)?** To do this we observe that, from the definition of \( v \), we have

\[
\mathcal{L} w = v. \tag{3.10}
\]

Thus

\[
w(x, t) = w_1(x, t) + \alpha(t) e^{\lambda_1 x}, \quad \text{where} \quad w_1(x, t) = \int_0^\infty G(x, y) v(y, t) \, dy, \tag{3.11}
\]

\( \alpha \) is a function to be determined, \( \lambda_1 \) is defined below, and \( G \) is the Green’s function for \( \mathcal{L}^{-1} \) with zero boundary condition at \( x = 0 \). That is:

\[
G = \frac{1}{\lambda_1 - \lambda_2} \left( e^{h(x-y)} - e^{\lambda_1 x - \lambda_2 y} \right) \tag{3.12}
\]

where \( h = \lambda_2 \) if \( x < y \), \( h = \lambda_1 \) if \( x > y \),

\[
\lambda_1 = -\nu - \sqrt{\nu^2 + (1 - \Omega^2)} < 0,
\]

and

\[
\lambda_2 = -\nu + \sqrt{\nu^2 + (1 - \Omega^2)} > 0.
\]

The \( \lambda_j \) are the two roots of \( \lambda^2 - 2\nu\lambda = 1 - \Omega^2 \), the characteristic equation for \( \mathcal{L} \).

**Why (3.11)?** Because \( \mathcal{L} w = v \) determines \( w \) up to an homogeneous solution, but \( e^{\lambda_2 x} \) is not allowed because \( \lambda_2 > 0 \).

Now, because \( v \) satisfies (3.1), and \( \mathcal{L} w_1 = v \),
we have \( \mathcal{L}((w_1)_{tt} - (w_1)_{xx} + w_1) = 0 \), hence

\[
(w_1)_{tt} - (w_1)_{xx} + w_1 = \beta(t) e^{\lambda_1 x}. \tag{3.13}
\]

But both \( v \) and \( w_1 \) vanish at \( x = 0 \). Hence evaluating (3.10) and (3.13) at \( x = 0 \) we obtain:

\[
(w_1)_{tt} - (w_1)_{xx} + w_1 = \beta(0) e^{\lambda_1 x}. \tag{3.14}
\]

Finally, substituting \( w = w_1 + \alpha(t) e^{\lambda_1 x} \) into \( w_{tt} - w_{xx} + w = 0 \), and using (3.13), yields an **equation that determines \( \alpha \).** That is

\[
\alpha = (1 - \lambda_2^2) \alpha + \beta = 0. \tag{3.15}
\]

The task of finding how to get initial conditions for this equation is left to the reader. Note that \( w \), as defined by all these steps, satisfies the boundary condition at \( x = 0 \). Why?

Because using \( w_{tt} - w_{xx} + w = 0 \) in \( \mathcal{L} w = v \) yields \( v = w_{tt} + \Omega^2 w - 2\nu w_x \), and \( v \) vanishes at \( x = 0 \).

**Remark 3.2** Provided that the initial data are reasonably smooth: as \( t \to \infty \), \( v \) vanishes; consequently, \( w \) as well.

It follows that: **As \( t \to \infty \), the solution to (3.1–3.3) is dominated by the particular solution, \( u \sim u_p \).***

**Here you may wonder:** wait a second, the particular solution is not unique; what if I use a different one from the one above?

The answer is that it does not matter: the difference between any two particular solutions vanishes.

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### 4 Transmission and reflection coefficients: 2-D wave equation #01

#### 4.1 Statement: Transmission and reflection coefficients: 2-D wave equation #01

Consider the wave equation in 2-D

\[
u_{tt} - \text{div}(c^2 \nabla u) = 0, \tag{4.1}
\]

with associated energy equation

\[
\left(\frac{1}{2} u_t^2 + \frac{1}{2} c^2 (\nabla u)^2 \right)_t - \text{div}(c^2 u_t \nabla u) = 0. \tag{4.2}
\]

Assume that

\[
c = c_1 \quad \text{for} \quad y > 0 \quad \text{and} \quad c = c_2 \quad \text{for} \quad y < 0, \tag{4.3}
\]

where \( c_1 \) and \( c_2 \) are positive constants. Across \( y = 0 \) we **require that both \( u \) and \( c^2 u_y \) be continuous.**

Calculate the **reflection** \( R \) and the **transmission** \( T \) coefficients\(^3\) for the plane wave, incident into the plane \( y = 0 \) from \( y > 0 \), given by

\[
\text{wave} = e^{i k \theta_i} \quad \text{where} \quad \theta_i = x \sin \phi - y \cos \phi - c_1 t. \tag{4.4}
\]

\(^3\) Of course, you cannot calculate \( T \) when the situation in item 2 applies. Then only \( R \) makes sense.
Here $k > 0$ is the wave-number, and $0 \leq \phi < \pi/2$ is the angle of incidence. Furthermore, obtain Snell’s law and

1. **Verify that $R$ and $T$ satisfy the appropriate energy budget:** The incident wave energy flux per unit length along the interface is equal to the sum of the transmitted and reflected fluxes.

2. If $c_1 < c_2$, there is a critical angle $\phi_c$ such that: for $\phi > \phi_c$ there can be no transmitted wave. **In this case, calculate** the reflection coefficient, and the evanescent wave on $y < 0$, and **verify that** $|R| = 1$ — as it should be because of energy conservation.

What happens with your answer when $\phi = \phi_c$? Does it make physical sense? What happens with the energy? **What conclusions can you draw?**

3. If $c_1 > c_2$, what happens with your answer as $\phi \to \pi/2$?

4. If $c_1 < c_2$, what happens with your answer as $\phi \to \pi/2$?

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**THE END.**