

Problem Set 1 solutions (by Joe Smith)

18.376 Wave propagation (MIT Spring 2023)

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Problem 1. The Allee effect (20 points)

For certain species of organisms, the effective growth rate \dot{N}/N is highest at intermediate N . This is called the Allee effect (Edelstein-Keshet, 1988). Imagine for instance that it is too hard to find mates when N is very small, and there is too much competition for food and other resources when N is large.

- (a) Show that $\dot{N} = f(N) = N(r - a(N - b)^2)$ provides an example of Allee effect, i.e. r , a and b satisfy certain constraints, to be determined.
- (b) Find all the fixed points of the system and classify their stability.
- (c) Sketch the solutions $N(t)$ for different initial conditions.
- (d) Compare the solutions $N(t)$ to those found for the logistic equation. Are there any qualitative differences?

Solution

Here it is convenient to have the solution in an itemized list, with the same item names as in the problem statement.

- (a) One must have $r > 0$, $a > 0$ and $b > 0$ in order to obtain a phase plot in the shape of Figure 1-(a).
- (b) Let us first enumerate the zeros of $\dot{N} = N(r - a(N - b)^2)$. There are three of them:

$$N_1 = 0, \quad N_2 = b - \sqrt{\frac{r}{a}}, \quad N_3 = b + \sqrt{\frac{r}{a}}. \quad (1)$$

Note that I added a “label” to the equation, so that I can refer to it in the text as follows: see equation (1). To see how this works, see the LaTeX file. Here is another example of an equation:

$$f'(N_1) = r - ab^2, \quad f'(N_2) = 2a\sqrt{\frac{r}{a}} \left(b - \sqrt{\frac{r}{a}} \right), \quad f'(N_3) = -2a\sqrt{\frac{r}{a}} \left(\sqrt{\frac{r}{a}} + b \right). \quad (2)$$

This is equation (2).

Note that you can have an itemized list within another. For example:

- b.1 Whatever you need to say here.
- b.2 Case (27) \Rightarrow case (28).
- (c) See Figure 1-(c-d).
- (d) There are qualitative differences with the logistic equation, which in each case are:
 - (i) One more fixed point than for the logistic equation.

(ii) The stability of $N = 0$ is different.

Of course, for many problems you will need to add figures. The pictures themselves must be stored in an image file, which can be pdf, png, jpeg, etc.

Below I show two examples of this.

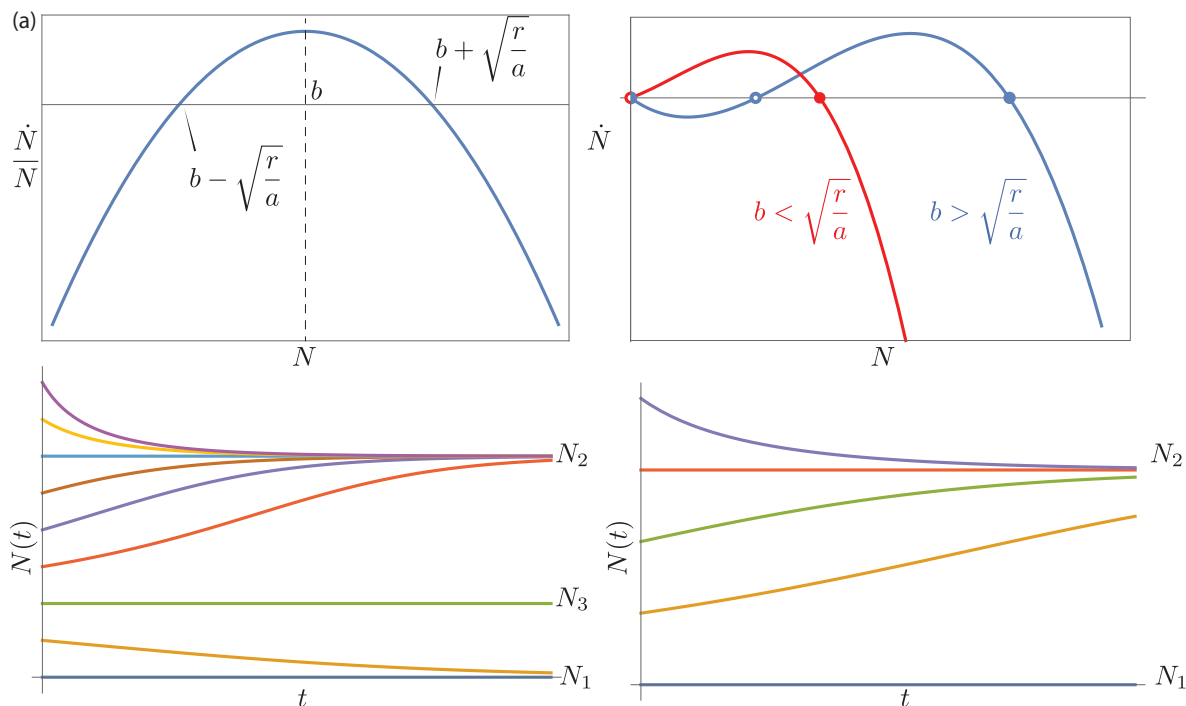


Figure 1: (problem 1). (a) Allee effect. (b) Phase plot of the problem. (c-d) Solutions in case 1 and 2, resp.

Here are some examples of things that you can do.

- **Spell-check your document.** This is not a "can do" but "must do". LaTeX packages include spellcheckers that are "LaTeX wise" (they do not mark LaTeX commands as errors). Use them.
- You can write equations within the text: $\sqrt{a^2 + b^2}$, $\frac{b}{1+x^2}$, $\left(1 + \frac{b}{1+x^2}\right)$ — note in the last one how to do "big" parenthesis; $\{x_n\}_{n=1}^{10}$ — note that LaTeX does not display brackets unless you precede them by a slash. You can also have the formulas in boldface $\frac{b}{1+x^2}$, $\left(1 + \frac{b}{1+x^2}\right)$, $\mathbf{a} + \mathbf{b} = \mathbf{2}$, or display them

$$\left(1 + \frac{b}{1+x^2}\right)$$

without a number, or as a numbered equation, as in equation (2).

- You can also **do colors**, with any font: **boldface**, or **serif**, or *italics*, etc. However, **avoid unreadable stuff** (e.g. light colors, yellow, green, etc.): **can you read this? Green is not so hot either!**
- Spaces are important in formulas. You can add extra space with various commands. Example: 1, 2, 3, 4, 5, 6; each adding more space than the prior. You can also combine/repeat them, as in: 1, 2, or 1, 2.

- **Important:** parentheses and the like: (u), {p}, [q], have to be done in “matched pairs”, or LaTeX can get “confused” and give you an error message [this is not 100% true, but till you know better, do so].
- You can add footnotes¹ anywhere you want.
- There are tons of other things that you can do, but for the problem sets the stuff here should be plenty good enough. But, if you need to do something, usually a Google [How do I do X in latex?] will lead you to 50 answers, one of them practical. For basic stuff, it is better to check the L^aT_EX manual. There are also tons of “tutorials” in the web.

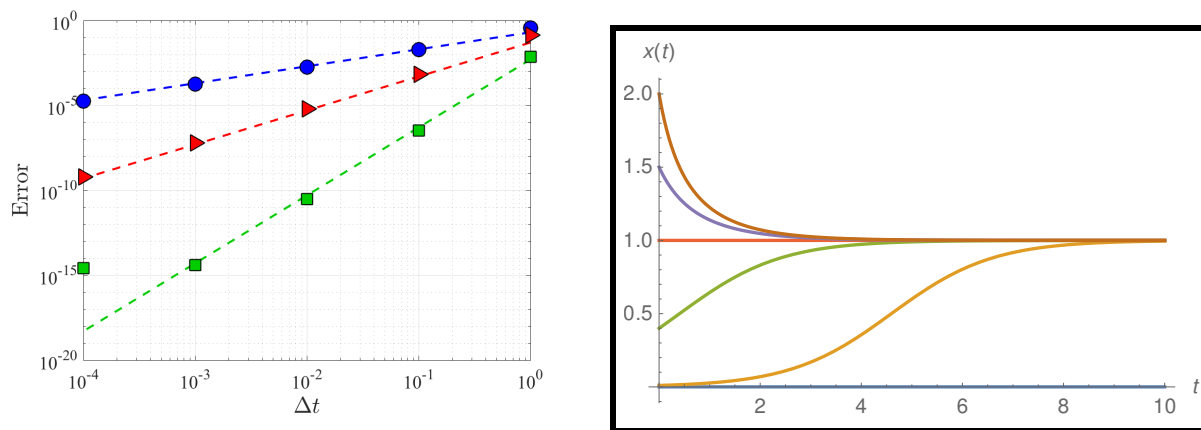


Figure 2: Another example. Note that here I included two pictures; the right one inside a box. By the way: **USE CAPTIONS**; do not include mystery pictures in your answers. Similarly: **Have labels on plot axes**.

¹This is a footnote