Stress Tensor

Definition: forces in a solid

1. Forces are given by a tensor
2. 2-D picture
3. Stress tensor is symmetric
4. Transformation properties
5. Principal axes

Note: Use the matrix notation in §5 from the word go.

\{ Pressure, shear force, decomposition of the stress tensor \}
Stress Tensor

Let $\mathbf{x}$ be an inertial, cartesian, coordinate system, and consider some substance/material under stress.

Define, for any element of surface $dS$ with unit normal $\hat{n}$ (see figure),

$$\mathbf{F} = \mathbf{F}(\hat{n}, \mathbf{x}) = \text{force per unit area across the surface}$$

by the same argument as in the problem "why is the flux a vector?"
\( \vec{F} \) is given by a tensor. Namely
\[
\vec{F} = \hat{n} \cdot \tau \quad \text{or} \quad F_n = \hat{n}_m T_{mn}
\]
where \( \tau = \tau(\vec{x}) \) is the stress tensor.

**Notation**

1. We use the repeated index summation convention.
2. The dot denotes tensor contraction, so that \( \hat{n} \cdot \tau = \hat{n}_j T_{ji} \) and \( \tau \cdot \hat{n} = T_{ji} \hat{n}_i \).
3. \( \nabla \) is the vector \( \frac{\partial}{\partial x_j} \).
4. \( \text{div} \tau = (T_{ji}) x_j \).
5. \( \tau^T \) is the transpose of \( \tau \).
Two-D picture illustrating the meaning of $T$

$T$ is a symmetric tensor $\mathbf{5}$

$T_{ij} = T_{ji}$

**Proof** Consider a sphere of radius $R$ centered at some arbitrary point $\mathbf{x}_0$. Denote the sphere by $\mathcal{S}$ and its boundary by $\partial\mathcal{S}$. The torque on the sphere is then

$\mathbf{T} = \int (\mathbf{n} \cdot \mathbf{T}) \times (\mathbf{x} - \mathbf{x}_0) \, dS =
\begin{cases}
\mathcal{S} & \text{if} \mathbf{n} \cdot \mathbf{T} > 0 \\
\mathcal{V} & \text{if} \mathbf{n} \cdot \mathbf{T} < 0
\end{cases}$

$= \int \nabla \cdot \left[ \mathbf{T} \times (\mathbf{x} - \mathbf{x}_0) \right] \, dV$
where
\[ \text{div} \left[ \mathbf{T} \times (\mathbf{x} - \mathbf{x}_0) \right] = \left( \text{div} \mathbf{T} \right) \times (\mathbf{x} - \mathbf{x}_0) \]
\[+ \left\{ T_{ij} \varepsilon_{gji} \right\} \]
Since \( \left( \mathbf{T} \times \mathbf{z} \right)_{il} = T_{ij} \varepsilon_{gji} m Z_m \)

However, in order to avoid unbounded angular accelerations, \( \mathbf{\dot{\Omega}} \) must behave like \( O(R^q) \) as \( R \rightarrow \infty \).
Hence \( T_{ij} \varepsilon_{gji} = 0 \) \( \forall \mathbf{E} \Rightarrow T_{ij} = T_{ji} \)

† If \( \mathbf{\ddot{a}} \) is the acceleration and \( \rho \) the mass density, then
\[ \int \rho \mathbf{\ddot{a}} \times (\mathbf{x} - \mathbf{x}_0) \, dV = \mathbf{\dot{T}} \]
\[ \frac{\partial}{\partial t} \]

Ignoring body forces and assuming \( \mathbf{\ddot{a}} \) is not changing
Transformation Properties

Let the two cartesian coordinates $\vec{x}_a$ & $\vec{x}_b$ be related by the orthogonal transform $T$. Namely

$$\vec{x}_b = T \vec{x}_a, \quad \text{Then } T_b = T T_a T^+ \quad (6)$$

Proof Use matrix notation, with vectors being column vectors and $T$ the matrix $\{T_{ij}\}$. Then $\equiv$

$$\begin{bmatrix} \vec{F} = T^+ \vec{\hat{n}} = T \vec{\hat{n}} \end{bmatrix} \quad (7)$$

$$T = T^+ \text{ from } (5)$$

Now $\vec{F}_b = T \vec{F}_a, \quad \vec{\hat{n}}_b = T \vec{\hat{n}}_a \Rightarrow$

$$\vec{F}_b = T \vec{F}_a = T T_a \vec{\hat{n}}_a = T T_a T^+ \vec{\hat{n}}_b$$
Principal Axes & physical meaning

At each point a rotation can be applied that makes $T$ diagonal. The coordinate axis where this happens are the principal axes. Then

\[ \text{Force} = T_{22} \]

\[ \text{Force} = T_{11} \]

\[
\begin{align*}
\text{Squeeze and pull, no shear} \\
\text{along sides of elementary cube of } \\
\text{side } dx
\end{align*}
\]
Pressure - shear decomposition of stress tensor

Let $p = -\frac{1}{3} \text{Trace } T = T_{nn}$

Then $T = -p I + T^*$ \( \text{(3)} \)

Where $T^*$ is symmetric with zero trace (hence the sum of the 3 principal forces by $T^*$ is zero)

$p$ is the pressure. If $T^* = 0$ then the stress is normal to any force, and independent of direction.

The response to pressure only affects by an isotropic material will be a volume change.

In elastic, isotropic, materials $p$ is caused by volume changes (same as for fluids).

On the other hand, if $p = 0$, the response of an isotropic material will be shape changes (with the same volume). In fluids, $T^*$ is produced by shear (viscous forces) while in elastic media it results from shear deformations.