4.0.1 Example for vector pde solutions

Here we deal with the following topic: In a vector complete orthonormal set of functions, each of the components is overpopulated (as explained below). **This then causes some counter-intuitive effects in vector normal mode solution expansions. For simplicity we use 2-vectors in 1-space dimensions.**

Consider a complete orthonormal set of 2-vector valued functions $\{\vec{\phi}_n(x)\}$, (4.1) VPS:eqn:01

$0 < x < 1$. Let the a_n and b_n be the components of $\vec{\phi}_n$, so that $\vec{\phi}_n = (a_n,\, b_n)$. (4.2) VPS:eqn:02

The orthonormality conditions are then
$$\delta_{n\,m} = \langle \vec{\phi}_n,\, \vec{\phi}_m \rangle_v = \langle a_n,\, a_m \rangle + \langle b_n,\, b_m \rangle, \quad (4.3)$$
VPS:eqn:03

where $\langle \vec{u},\, \vec{v} \rangle_v$ is the vector inner product,
while $\langle f,\, g \rangle = \int_0^1 f(x)\, \overline{g}(x)\, \mathrm{d}x$ denotes the standard inner product for scalar functions.

It follows that, for any square integrable vector function $\vec{u} = \vec{u}(x)$,
$$\vec{u} = \sum_n u_n\, \vec{\phi}_n, \quad (4.4)$$
VPS:eqn:04

where $u_n = \langle \vec{u},\, \vec{\phi}_n \rangle$. Component-wise,
with $\vec{u} = (\alpha,\, \beta)$, this is
$$\alpha = \sum_n (\alpha_n + \beta_n)\, a_n(x) \quad \text{and} \quad \beta = \sum_n (\alpha_n + \beta_n)\, b_n(x), \quad (4.5)$$
VPS:eqn:05

where $\alpha_n = \langle \alpha,\, a_n \rangle$
and $\beta_n = \langle \beta,\, b_n \rangle$.

Weird thing #1. (4.5) shows two different (arbitrary) functions, α and β, expanded in terms of two different[†] function sets, $\{a_n\}$ and $\{b_n\}$, using the **same coefficients** $u_n = \alpha_n + \beta_n$ **How is this possible?**

† If it was $a_n = b_n$, then it would be $\alpha = \beta$. At the very least the two sets have to be ordered differently.

Weird thing #2. Now we explain "weird thing #1" using another weird thing. Since α_n depends on α only (and β_n on β only), apply (4.4–4.5) to
$\vec{u} = (\alpha,\, 0)$ and $\vec{u} = (0,\, \beta)$, and conclude
$$0 = \sum_n \beta_n\, a_n(x) \quad \text{and} \quad 0 = \sum_n \alpha_n\, b_n(x). \quad (4.6)$$
VPS:eqn:06

Then
$$\alpha = \sum_n \alpha_n\, a_n(x) \quad \text{and} \quad \beta = \sum_n \beta_n\, b_n(x), \quad (4.7)$$
VPS:eqn:07

which removes the "how can we have the same coefficients?" question, but opens up a deeper question: **How is (4.6) possible?** (4.8) VPS:eqn:08

Explanation in remark 4.1. **Important: see remark 4.2!**

Remark 4.1 On the size of complete sets. Consider a finite dimensional vector space V. Then a base $\{\vec{x}_n\}$ has $N = \text{dimension}(V)$ elements. On the other hand, a base $\{\vec{\phi}_n = (\vec{a}_n, \vec{b}_n)\}$ for $V \oplus V$ (the set of all $(\vec{x},\, \vec{y})$ with $\vec{x},\, \vec{y} \in V$) has $2\,N$ elements. This means that $\{a_n\}$ and $\{b_n\}$ have twice as many elements as needed for a base; thus (though complete sets) they are **not linearly independent.** When going to **infinite dimensions** we can no longer talk about "twice as many", but **it remains true that $\{a_n\}$ and $\{b_n\}$ in (4.2) are not linearly independent.** Hence **there is nothing strange about (4.6).**

Example 1: Consider the operator[a] ∂_x^2 on 2-vector valued functions on $[0,\, \pi]$, vanishing at the end points. This has double eigenvalues $\{-n^2\}$, with eigenfunctions $\vec{\phi}_n = (\sin(n\,x),\, 0)$ and $\vec{\psi}_n = (0,\, \sin(n\,x))$. This corresponds to the **trivial situation:** "half" the $\{a_n\}$ vanish, and the others form an orthonormal set (with the $\{b_n\}$ doing the reverse, and vanishing exactly when the $\{a_n\}$ do not). Then (4.6) follows trivially.

Example 2: Consider[b] $\mathcal{L}(f,\, g) = (g_x,\, f_x)$ on 2-vector valued functions on $[0,\, \pi]$, with $f = 0$ at the end points. This has eigenvalues $\{-i\,n\}$, with eigenfunctions $\vec{\phi}_n = (\sin(n\,x),\, i\,\cos(n\,x))$. Then $\sin(n\,x)$ shows up twice in the $\{a_n\}$, and $\cos(n\,x)$ twice in the $\{b_n\}$. So, again, no problem with (4.6).

a This operator is self-adjoint and negative definite. *b* This operator is skew-adjoint ♣

Remark 4.2 **Implications for pde solutions.** Suppose now that the set in (4.1) is the set of eigenfunctions for the operator $\mathcal{L}$ in a pde $\vec{u}_t = \mathcal{L}\,\vec{u}$. Then the solution to the pde with initial values $\vec{u}(0,\, t) = (\alpha,\, \beta)$ is $u = \sum u_n\, e^{s_n t}\, \vec{\phi}$, where the $\{s_n\}$ are the eigenvalues.

Component-wise
$$u_1 = \sum_n (\alpha_n + \beta_n)\, e^{s_n t}\, a_n(x) \quad \text{and} \quad u_2 = \sum_n (\alpha_n + \beta_n)\, e^{s_n t}\, b_n(x). \tag{4.9}$$
VPS:eqn:09

Question: **does (4.6)**
mean that we can write $u_1 = \sum_n \alpha_n\, e^{s_n t}\, a_n(x)$ **and** $u_2 = \sum_n \beta_n\, e^{s_n t}\, b_n(x)$**?**
Absolutely no!
(4.6) implies neither $0 = \sum_n \beta_n\, e^{s_n t}\, a_n(x)$ **nor** $0 = \sum_n \alpha_n\, e^{s_n t}\, b_n(x)$. This is true only for $t = 0$.
Intuition: For t small
the first component of the solution is dominated by the initial data for the first component (and similarly for the second). But after a while the interaction (generally) will both sets of initial data be important for both component. **This behavior should be expected, so that (in fact) (4.6) is not only not "strange", but in fact what intuition dictates it should happen!** ♣