Moving point source in 2d
(and Kelvin ship wake angle)

R. R. Rosales (MIT, Math. Dept.). May 14, 2023

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1 Moving point source in 2d (and Kelvin ship wake angle)

We consider a *point source in a 2D linear dispersive system, moving at constant velocity* $\vec{V}$. W.L.O.G. assume: **the source is at $x_1 = V\,t$ and $x_2 = 0$, with $V > 0$.**

In particular, we concentrate on problems of the form
$$\mathcal{L}u = \delta(x_1 - V\,t)\delta(x_2), \tag{1.1}$$
where $\mathcal{L}$ is the "dispersive wave operator" (see note 1.2).

Alternatively, for some f decaying at infinity (see note 1.1)
$$\mathcal{L}u = f(x_1 - V\,t,\, x_2), \tag{1.2}$$
We will **not** consider problems of the form
$$\mathcal{L}u = \delta(x_1 - V\,t)\delta(x_2)\, e^{i\,\mu\,t}, \tag{1.3}$$
or, more generally,
$$\mathcal{L}u = f(x_1 - V\,t,\, x_2,\, t), \tag{1.4}$$
The solution to **(1.2) is a linear superposition** of **solutions to problems of the form (1.1),** while (1.4) is linked in the same fashion to (1.3).

- If u solves (1.1) then $\int u(x_1 - y_1,\, x_2 - y_2,\, t)\, f(y_1,\, y_2)\, \mathrm{d}y_1\, \mathrm{d}y_2$ solves (1.2).
- In (1.4) let $f(x_1,\, x_2,\, t) = \int g(x_1,\, x_2,\, \mu)\, e^{i\,\mu\,t}\, \mathrm{d}\mu$.
  Then $\int u(x_1 - y_1,\, x_2 - y_2,\, t,\, \mu)\, g(y_1,\, y_2,\, \mu)\, \mathrm{d}y_1\, \mathrm{d}y_2\, \mathrm{d}\mu$ solves (1.4) if u solves (1.3).

Note 1.1 Far field limit. In case (1.2) we **consider only the "far field limit",** where the source looks like a point. This problem is more general than (1.1), because, through the Fourier Transform $\hat{f}$, it allows for a generic power spectrum in the source, while (1.1) has the same energy across the spectrum. ♣

Note 1.2 Why only scalar problems? The problems listed above are scalar. This is not a limitation: any problem can be decomposed as a linear superposition of the response of each dispersive branch $\omega = \Omega^{(j)}(\vec{k})$, and each branch acts as a scalar problem (amplitude of the corresponding eigen-direction). The question is then how much energy goes into each dispersive mode — characterized by a $(\vec{k},\, \Omega^{(j)}(\vec{k}))$. This is a nontrivial problem we will not discuss. For example: in the case of a ship wave pattern, the "near source problem" is highly nonlinear, and depends on ship details as well as the local sea conditions. ♣

We will *tackle the problem in two alternative ways;* via: (1) **modulation theory;** and (2) **Fourier transform and Duhamel's principle**.

1.1 The modulation theory approach

Justification. Let ϵ be the **ratio of the source size to the observation distance:** $0 < \epsilon \ll 1$ in the far field limit. Because different waves propagate at different speeds, and they all originate at the source, in the far field each location corresponds to one value of the **wave vector** $\vec{k}$ (or, at most, a small discrete set of values).

Further: this value changes slowly and smoothly with location almost everywhere (at least if the group speed is a smooth function of $\vec{k}$). This is exactly the set-up where modulation theory applies **(see § 1.4.** The only exceptions to this are things like caustics, and arêtes, where new wave numbers appear or disappear (in pairs at caustics). However, we know how to deal with these objects within the context of modulation theory. ♣

Here we will show that the **2-D far field, steady state wave pattern,** for an **homogeneous linear dispersive system** with **dispersion function** $\Omega = \Omega(\vec{k})$ (here $\vec{k} = (k_1,\, k_2)$ is the wave-vector), produced by a **source traveling at constant speed** $V > 0$ along the x_1 axis (i.e.: $x_1 = V\,t$ and $x_2 = 0$), is characterized by

$$\left.\begin{array}{lrcl} \textbf{[A]} & \Omega & = & k_1\,V, \text{ with } \textbf{[B]}\ \vec{k} = \vec{k}(\phi), \\ \textbf{[C]} & 0 & = & \Omega_2 \cos\phi - (\Omega_1 - V)\sin\phi, \\ \textbf{[D]} & 0 & < & (\Omega_1 - V)\cos\phi + \Omega_2 \sin\phi, \end{array}\right\} \text{ and } \left.\begin{array}{lrcl} \textbf{[E]} & \theta & = & \frac{1}{\epsilon}(k_1\,x + k_2\,y + \Theta_0), \\ \textbf{[F]} & A & = & \frac{1}{\sqrt{r\,\kappa}}\,A_0(k_2), \end{array}\right\} \quad (1.5)$$

where: **(I)** $\vec{k}$ and the (complex) **wave amplitude** A are functions of $x = \epsilon\,(x_1 - V\,t) = r\,\cos\phi$ and $y = \epsilon\,x_2 = r\,\sin\phi$. **(II)** $\vec{c}_g = (\Omega_1 - V,\, \Omega_2)$ is the **group speed in the moving frame**, with[1] $\Omega_n = \partial_{k_n}\Omega$ — if the dispersion relation has multiple branches, each must be considered separately (see note 1.2). Generally, at least two expressions like (1.6) appear, since the complex conjugate is needed. **(III)** The variables are adimensional: x_j is scaled by the source size and t so that $V = O(1)$. **(IV)** θ is the **wave phase**, and Θ_0 is a constant. **(V)** $\kappa = \left|\frac{d\phi}{dk_2}\right|$, where ϕ is defined as a function of k_2 by **[B]**. **(VI)** A_0 characterizes how the energy input from the source is distributed amongst the wave-numbers. This function cannot be obtained through modulation theory techniques alone. **(VII)** Note that **[F]** gives an infinite amplitude as $r \to 0$. However, this is a far field approximation, and it is not meant to be valid near the source. In fact, from the far field the source looks like a Dirac delta function, so an infinity as $r \to 0$ is to be expected. The **theory is then applied to deep gravity waves,** to obtain the **Kelvin ship wave pattern** in § 1.2.

We **seek a solution** that is a superposition of slowly varying waves where $\tau = \epsilon\,t$, and $\theta = \frac{1}{\epsilon}\Theta(x,\, y,\, \tau)$.

$$u \sim A(x,\, y,\, \tau)\,e^{\theta}, \quad (1.6)$$

Then

$$(k_1)_\tau + (\Omega - k_1\,V)_x = 0,\ (k_2)_\tau + (\Omega - k_1\,V)_y = 0,\ (k_1)_y - (k_2)_x = 0, \quad (1.7)$$

and

$$(A^2)_\tau + ((\Omega_1 - V)\,A^2)_x + (\Omega_2\,A^2)_y = 0, \quad (1.8)$$

where **(I)** $k_1 = \Theta_x$ and $k_2 = \Theta_y$ are the components of the **wave-vector**. **(II)** $-\Theta_\tau = \Omega - k_1\,V$ is the **angular frequency in the moving frame**. **(III)** $\Omega - k_1\,V$ is the **dispersion function in the moving frame** (in general, the **angular frequency transforms to a moving frame via** $\omega \to \omega - \vec{k}\cdot\vec{V}$).

Only the **steady state pattern, where $\Theta = \Theta(x,\, y)$** — so that:

$$\Omega = k_1\,V, \quad (1.9)$$

is of interest to us here. Then:

$$\text{(a)}\ \ (k_1)_y - (k_2)_x = 0, \ \text{ and }\ \text{(b)}\ \ ((\Omega_1 - V)\,A^2)_x + (\Omega_2\,A^2)_y = 0, \quad (1.10)$$

with the first two equations in (1.7) trivially satisfied. Note, (1.9) implies: **k_1 and Ω have the same sign,** consistent with the fact that, at steady state, **all the wave-crests move left to right, in lockstep with the source.**[♭]

♭ The geometrical picture is as follows: imagine a wave-crest (constant phase straight line). Then its intersection with the x_1 axis must move at speed V to keep the solution steady. This is all (1.9) states.

As expected, these equations can be

[1] This notation should not be confused with the notation for the dispersion branches: $\Omega^{(j)}$.

formulated in terms of **wave rays**,† $$\frac{d}{d\tau}(x,\, y) = \vec{c}_g = (\Omega_1 - V,\, \Omega_2). \quad (1.11)$$

† At steady state there is no time, and we can select an arbitrary parameter, s, along the wave rays; i.e.: $\frac{d}{ds}(x,\, y) = \alpha\, \vec{c}_g$ works as long as α does not vanish anywhere. However, when $\alpha = 1$, s is time, and **corresponds to the flow of information along the wave ray — important, because the steady state solution is maintained by the continuous flow of information from the source.**

In fact, in (1.10a) we can think of k_2 as a function of k_1, via (1.9), and write
$$\frac{d}{dy}\, k_1 = 0 \quad \text{along} \quad \frac{dx}{dy} = -\frac{dk_2}{dk_1} = \frac{\Omega_1 - V}{\Omega_2}, \quad (1.12)$$
where $\frac{dk_2}{dk_1}$ follows from (1.9). This shows that [2] **$\vec{k}$ is constant on rays, which are thus straight lines** (since $\Omega_n = \Omega_n(\vec{k})$). But the solution is generated by the source, hence **the rays are straight lines through the origin.** In polar coordinates $x = r\, \cos\phi$ and $y = r\, \sin\phi$
$$\vec{k} = \vec{k}(\phi), \quad (1.13)$$
with the restriction ‡
$$\Omega_2 \cos\phi = (\Omega_1 - V)\sin\phi. \quad (1.14)$$

‡ Equation (1.13) is, roughly, only the first equation in (1.12). For a solution both must be satisfied. To obtain (1.14), note that in the second equation in (1.12), $dx = \cos\phi\, dr$ and $dy = \sin\phi\, dr$. **Alternatively:** look for a solution of (1.9) and (1.10a) of the form in (1.13).

Equation (1.14) states that $\vec{c}_g = (\Omega_1 - V, \Omega_2)$ is aligned with the radial direction $\vec{r} = (x,\, y)$, but it **does not include causality:** the waves originate at the source, hence **$\vec{c}_g$ must be a positive multiple of $\vec{r}$.** i.e.:
$$(\Omega_1 - V)\cos\phi + \Omega_2 \sin\phi > 0. \quad (1.15)$$
Equations (1.9) and (1.13–1.15), the same as **[A–D]** in (1.5), **characterize the far field distribution of wave-vectors in space.** Note also that
$$\Theta = k_1\, x + k_2\, y + \Theta_0, \quad (1.16)$$
where Θ_0 is a constant. To check this note that, after use of (1.10a), $\Theta_x = k_1 + x\,(k_1)_x + y\,(k_1)_y$. But then (1.13) yields $\Theta_x = k_1$. Similarly $\Theta_y = k_2$.

Next we turn our attention to (1.10b), to **characterize the wave amplitude.** Consider the "ray-strip", given by $\phi_1 < \phi < \phi_2$ and $r_1 < r < r_2$, and the integral of the divergence of $\vec{F} = \vec{c}_g\, A^2$ on the strip. Then Gauss theorem and (1.10b) yield
$$\int_{\phi_1}^{\phi_2} (c_g\, A^2)_{r=r_1}\, r_1\, d\phi = \int_{\phi_1}^{\phi_2} (c_g\, A^2)_{r=r_2}\, r_2\, d\phi, \quad \textbf{[In]}$$
where $c_g = |\vec{c}_g|$ (recall $\vec{c}_g$ points along the radial direction, away from the origin). Since the ϕ_n and r_n here are arbitrary, we conclude that $c_g\, A^2\, r$ is a function of ϕ only. But c_g is also a function of ϕ only. Hence $A = \alpha(\theta)/\sqrt{r}$, where α is a function that characterizes how the energy input from the source is distributed amongst the angles. However, an expression for A in terms of a **function A_0 that characterizes how the energy input from the source is distributed amongst the wave-numbers** would be more useful. To do this all we need to do is change the integration variable in **[In]** from ϕ to k_2, and repeat the argument.† This yields
$$A = \frac{1}{\sqrt{r\,\kappa}}\, A_0(k_2), \quad (1.17)$$
where $\kappa = \left|\frac{d\phi}{dk_2}\right|$.

† From (1.9–1.13) it should be clear that the rays can be parameterized by either ϕ or one of the k_j.

1.2 Example: deep gravity waves and the Kelvin ship wave pattern

Let g be the acceleration of gravity and $k = |\vec{k}| = \sqrt{k_1^2 + k_2^2}$. Then
$$\Omega = \sqrt{g\, k}, \quad (1.18)$$
where we consider only the positive branch, $\Omega \geq 0$. The full pattern follows from taking the real part of (1.6).

[2] "Horizontal" rays, corresponding to $\Omega_2 = 0$, require parameterization by x instead of y, but the conclusion is the same.

k01 Since the variables are adimensional, V and g are "place-holders" for the dimensional quantities (to track their effect on the pattern). We could have set either $V = 1$ or $g = 1$, but we did not — for the reason in item k02.

k02 **Re-scale** $\vec{k}$ and Ω as follows: $\vec{k} = \frac{g}{V^2}\,\vec{k}_\sharp$ and $\Omega = \frac{g}{V}\,\Omega_\sharp$. Then $\Omega = \sqrt{k}$, (1.19)
where we **drop the subscript $\sharp$ for notational simplicity,** while (1.5) becomes

$$\left.\begin{array}{llcl} \textbf{[A]} & \Omega & = & k_1, \text{ with } \textbf{[B]}\ \vec{k} = \vec{k}(\phi), \\ \textbf{[C]} & 0 & = & \Omega_2\cos\phi - (\Omega_1 - 1)\sin\phi, \\ \textbf{[D]} & 0 & < & (\Omega_1 - 1)\cos\phi + \Omega_2\sin\phi, \end{array}\right\} \text{ and } \left.\begin{array}{llcl} \textbf{[E]} & \theta & = & \frac{1}{\epsilon}(k_1\,x + k_2\,y + \Theta_0), \\ \textbf{[F]} & A & = & \frac{1}{\sqrt{r\,\kappa}}\,A_0(k_2), \text{ with} \\ & \kappa & = & |\mathrm{d}\phi/\mathrm{d}k_2| \end{array}\right\} \quad (1.20)$$

where we also implement $x = \frac{V^2}{g}\,x_\sharp$ and $y = \frac{V^2}{g}\,y_\sharp$. In these equations both V and g are gone which means that:

The pattern shape is independent of both the source velocity and gravity. (1.21)

The same for a duck or a yacht, on Earth or Mars, as long as only deep gravity waves are involved.

Pattern shape invariance is a property of dispersion functions where $\Omega \propto k^\beta$.

Important: "pattern" here means the wave vector distribution in space. **What we actually see depends on A_0 as well**, which is source dependent (if little energy goes into a wave-number, we will not see it).

k03 (1.19-A) is equivalent to $k_1^4 = k_1^2 + k_2^2$, with $k_1 \geq 0$. This yields (1.22), where in [c] the positive root is selected so that the right hand side is positive. Note that (1.22c) yields $k_1 \sim 1 + \frac{1}{2}\,k_2^2$ for k_2 small, and $k_1 \sim \sqrt{|k_2|}$ for $|k_2|$ large.

$$\left.\begin{array}{llcl} [a] & k_2 & = & \pm k_1\sqrt{k_1^2 - 1}. \\ [b] & k_1 & \geq & 1. \\ [c] & k_1^2 & = & \frac{1}{2}\left(1 + \sqrt{1 + 4\,k_2^2}\right). \end{array}\right\} \quad (1.22)$$

k04 In item k03 we have **excluded the root $k_1 = k_2 = 0$.** This root is isolated, and thus cannot be part of a slowly varying wave pattern. It also has infinite group and phase speeds. These unrealistic speeds arise from the fact that, for sufficiently long waves the "deep fluid" approximation (i.e.: $k\,h \gg 1$, h = fluid depth) breaks down. It corresponds to a constant that can be added to (1.6) — which does not affect the pattern.

k05 The range $0 < k_1^2 < 1$ corresponds to k_2 purely imaginary, and probably becomes significant when the source is dragged along a wall (i.e.: wall at $x_2 = 0$). Then the pattern would include edge waves. I do not know, I have not checked this situation

k06 The group speed components are given by $\Omega_1 - 1 = \frac{1}{2\,k_1^2} - 1$ and $\Omega_2 = \frac{1}{2\,k_1^3}\,k_2$. (1.23)
Thus **$-1 < \Omega_1 - 1 \leq -1/2$ is an even function of k_2**, while **Ω_2 is an odd function of k_2, positive for $k_2 > 0$**, with a maximum at $k_2 = \sqrt{2}$, where $\Omega_2 = 1/4$ and $k_1 = \sqrt{2}$. For $k_2 \gg 1$, $\Omega_2 \sim \frac{1}{2\sqrt{k_2}}$.

Proof. Direct differentiation of $\Omega = \sqrt[4]{k_1^2 + k_2^2}$ yields $\Omega_1 = \frac{k_1}{2\,\Omega^3}$ and $\Omega_2 = \frac{k_2}{2\,\Omega^3}$. Substituting (1.20A) then gives (1.23). The stated properties of $\Omega_1 - 1$ and Ω_2 follow from (1.22). In particular, note that $\Omega_2^2 = \frac{\mu - 1}{4\,\mu^2}$, where $\mu = k_1^2$, from which the maximum at $k_1 = \sqrt{2}$ follows. **QED**.

k07 Equations (1.20C-D) state that the group velocity has to be a positive multiple of the radial unit vector $(\cos\phi,\ \sin\phi)$. Thus, from† (1.23), we conclude: **(i)** There are no waves ahead of the source, $|\phi| < \pi/2$. **(ii)** $k_2 > 0$ in the second quadrant, $\pi/2 \leq \phi < \pi$. **(iii)** $k_2 < 0$ in the third quadrant, $-\pi \leq \phi < -\pi/2$. **(iv)** $k_2 = 0$, hence $k_1 = 1$, for $\phi = \pi$.

† The key here are the signs of the components of $\vec{c}_g$.

k08 Introduce $\psi = \pi - \phi$, i.e.: **measure angles from the negative x_1 axis.** Then, from item k07, equation (1.20D) is satisfied if we **restrict $|\psi| < \pi/2$**, while equation (1.20C) becomes

$$\tan\psi = \frac{k_2}{2\,k_1^3 - k_1}. \qquad (1.24)$$

Solving this for k_2 (or k_1) as a function of ψ provides (1.20B). Notice that the denominator here is always positive because $k_1 \geq 1$.
Proof. From (1.20D) $\frac{\sin\phi}{\cos\phi} = \frac{\Omega_2}{\Omega_1 - 1}$. Then use (1.23), $\sin\phi = \sin\psi$ and $\cos\phi = -\cos\psi$. **QED.**

k09 Clearly, in (1.24), $k_2 \to -k_2$ corresponds to $\psi \to -\psi$ **(reflection symmetry of the pattern).** Hence we can restrict the equation to $k_2 \geq 0$ and $0 \leq \psi \leq \pi/2$, where it is equivalent to its square. i.e.:

$$\tan^2\psi = \frac{k_1^2 - 1}{(2\,k_1^2 - 1)^2}, \qquad (1.25)$$

as follows from (1.22a).
In (1.25) $k_1^2 \geq 1$, and the right hand side has its maximum, $1/8$, at $k_1^2 = 3/2$. Hence

$$\text{The wave pattern is constrained to the wedge } |\psi| \leq \psi_c = \arctan(1/\sqrt{8}) \approx 19.47^o. \qquad (1.26)$$

For each $0 < |\psi| < \psi_c$ two wave numbers k_1 arise; except at $\psi = \pm\psi_c$, where there is only one: $k_1 = \sqrt{3/2}$ (corresponding to $k_2 = \pm\sqrt{3}/2$). Thus **the wave pattern is a superposition of two expressions of the form in (1.6).** For what happens near ψ_c, see item k12.

Note: as pointed out in (1.21), what is actually seen depends on A_0 as well.

k10 Equation (1.25) has the explicit solution in (1.27), where $\mu = 1/\tan^2\psi \geq 8$.

$$\begin{aligned} k_1^2 &= \tfrac{1}{8}\left(\mu + 4 \pm \sqrt{(\mu+4)^2 - 16\,\mu - 16}\right) \\ &= \tfrac{1}{8}\left(\mu + 4 \pm \sqrt{\mu^2 - 8\,\mu}\right). \end{aligned} \qquad (1.27)$$

k11 Equation (1.27) yields, **as $\mu \to \infty$ (i.e.: $\psi \to 0$), $k_1^2 = \frac{1}{8}\left((\mu+4) \pm (\mu - 4)\right) + O(1/\mu)$.** The $-$ sign root approaches the $\phi = \pi$ solution in item k07(iv). On the other hand, the $+$ sign root has $k_1^2 \sim \frac{1}{4}\,\mu$ (hence $k_2 \sim \pm\frac{1}{4}\,\mu$). This is a high frequency wave,† with the wave fronts nearly parallel to the x_1-axis, as needed to keep them stationary (i.e.: $(\vec{c}_p)_1 = V$, where $\vec{c}_p$ is the phase velocity). Note that: (i) We expect very little energy will go into these high frequencies. (ii) For short enough wave-lengths the gravity wave approximation breaks down, as then surface tension cannot be ignored.

† Note that high frequency gravity waves move very slowly: small group speed and phase speed.

k12 Finally, we **focus on the (complex) wave amplitude A,** given by (1.20F). We do not know A_0, so there is not much explicit that we can say about A. However, notice that (1.25) implies that $\kappa = |\mathrm{d}\psi/\mathrm{d}k_2|$ vanishes at $\psi = \pm\psi_c$, leading to the **prediction of an infinite amplitude for** $\psi = \pm\psi_c$, unless A_0 vanishes there. Of course, this is **precisely where (1.6) fails,**‡ because two wave vectors $\vec{k}$ merge and vanish across $\psi = \pm\psi_c$ — a **turning point (transition from waves to no waves).**

‡ That is: the theory in § 1.4 **fails** at turning points, but **it is known how to fix it.**

What happens is: **near $\psi = \pm\psi_c$ the amplitude decay rate is $1/\sqrt[3]{r}$**, not $1/\sqrt{r}$, with a **transition region characterized by an Airy function** — this is standard stuff, but I will not do it here.

1.3 Example: capillary waves

Not yet typed.

1.4 Review of modulation theory for homogeneous media

This theory generalizes the WKBJ method to dynamical (linear) wave problems. It assumes a slowly varying wave of the form

$$u \sim A(\vec{X},\, T)\, e^{\theta}, \qquad (1.28)$$

where $\vec{X} = \epsilon\,\vec{x}$, $T = \epsilon\, t$, and $\theta = \frac{1}{\epsilon}\Theta(\vec{X},\, \vec{T})$ — for some small $0 < \epsilon \ll 1$. Of course, appropriate adimensional variables are required. In our case x is scaled by the source size, and t is such that the source velocity V is $O(1)$. Given this, the following equations apply [3]

$$\text{(a) } \vec{k}_T + \nabla\,\omega = 0, \quad \text{(b) } (k_2)_{X_1} - (k_1)_{X_2} = 0, \quad \text{and} \quad \text{(c) } (A^2)_T + \operatorname{div}(\vec{c}_g\, A^2) = 0, \qquad (1.29)$$

where **$\vec{k} = \nabla\,\Theta = (\Theta_{X_1},\, \Theta_{X_2})$ is the wave-vector, $\omega = -\Theta_T = \Omega(\vec{k})$ is the angular frequency, Ω is the dispersion function, $\vec{c}_g = \vec{c}_g(\vec{k}) = \nabla_k\,\Omega = (\Omega_1,\, \Omega_2)$ is the group speed,** (a) and (b) state the "conservation of waves", and (c) is equivalent to the conservation of wave energy. **Notation:**

- **n1** Here $\Omega_1 = \partial_{k_1}\Omega$ and $\Omega_2 = \partial_{k_2}\Omega$. If the dispersion relation has multiple branches, each is considered separately (see note 1.2). Generally, at least two expressions like (1.28) appear, since the complex conjugate is needed.
- **n2** ∇ is the gradient with respect to the space variables, while ∇_k is the gradient with respect to $\vec{k}$.

Using (1.29b) we can re-write (1.29a), and obtain the alternative form:

$$\text{(a) } \vec{k}_T + (\vec{c}_g \cdot \nabla)\,\vec{k} = 0, \quad \text{(b) } (k_2)_{X_1} - (k_1)_{X_2} = 0. \qquad (1.30)$$

This explicitly shows the wave parameter propagation along **wave rays**, i.e.:

$$\frac{d}{dT}\,\vec{k} = 0 \quad \text{along} \quad \frac{d}{dT}\vec{X} = \vec{c}_g. \qquad (1.31)$$

Since $\vec{c}_g = \vec{c}_g(\vec{k})$, $\vec{c}_g$ is constant along each wave ray.
Thus the wave rays are straight lines. Note also that (1.30b) *is satisfied if it is satisfied at any T along the ray* † (e.g.: at the initial data, $T = 0$).

† See (1.34). H symmetric $\Rightarrow K$ and K^T satisfy $\frac{d}{dT}\,Y + Y\,H\,Y = 0$. Hence $K = K^T$ at any T yields $K \equiv K^T$.

The equation for A can be written in the form

$$A_T + (\vec{c}_g \cdot \nabla)\,A = -\tfrac{1}{2}\operatorname{div}(\vec{c}_g)\,A. \qquad (1.32)$$

Equivalently

$$\frac{d}{dT}\,A = -\tfrac{1}{2}\operatorname{div}(\vec{c}_g)\,A \quad \text{along} \quad \frac{d}{dT}\vec{X} = \vec{c}_g. \qquad (1.33)$$

For (1.33) to be useful, we need the evolution of $\operatorname{div}(\vec{c}_g)$ along rays. This follows from

$$\frac{d}{dT}\,K + K\,H\,K = 0, \qquad (1.34)$$

and

$$\frac{d}{dT}\,C + C\,H\,K = 0, \qquad (1.35)$$

where $H = \{\partial_{k_m}\partial_{k_n}\Omega\}$ is the dispersion function Hessian, and $(K,\, C)$ are the matrices with entries $K_{mn} = \partial_{X_n}\,k_m$ and $C_{mn} = \partial_{X_n}(\partial_{k_m}\Omega) = \partial_{X_n}(c_g)_m$. **Then $\operatorname{div}(\vec{c}_g)$ is the trace of C.**

Proof of (1.34). Take ∂_{X_n} of (1.30a), all n. This yields $K_T + (\vec{c}_g \cdot \nabla)\,K + K\,H\,K = 0$, which is (1.34).
Proof of (1.35). Since $\vec{c}_g = \vec{c}_g(\vec{k})$, $(\vec{c}_g)_T + (\vec{c}_g \cdot \nabla)\,\vec{c}_g = 0$ follows from (1.30a) — i.e.: $\vec{c}_g$ is constant along wave rays. Now use the same argument used to prove (1.34) to obtain (1.35).

1.5 The Fourier transform and Duhamel's principle approach

Not yet typed.

The End.

[3] In 3-D (1.29b) is replaced by $\nabla \times \vec{k} = 0$.