Average Lagrangian, adiabatic invariants, and related topics

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Abstract

The bulk of these lectures is inspired by the "Linear and Nonlinear Waves" book by G. B. Whitham [2].

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1 Slowly varying traveling waves

In this section we show how to represent (mathematically) **linear slowly varying traveling waves,** and then use the **variational principles that govern conservative systems** to obtain equations for the wave parameters. In particular, we derive the equation for the wave action.

1.1 Wave representation and Average Lagrangian

The first question is *How do we represent a slowly varying wave?* For **simplicity we will consider here scalar waves only,** $u = u(\vec{x}, t)$ — generalizations to vector valued waves are fairly straightforward.

Assume a-dimensional variables, where the length/time scales correspond to a typical wavelength/wave period. Then, on $O(1)$ regions in space-time, the wave is (approximately) a plane wave: $u \approx a \cos\theta$, where $\theta = \vec{k}\cdot(\vec{x}-\vec{x}_0) - \omega\,(t-t_0) + \theta_0$, with wave frequency and wave number satisfying a **dispersion relation** $G(\omega, \vec{k}) = 0$. However, over large, $O(1/\epsilon)$, regions $(0 < \epsilon \ll 1)$ the wave parameters vary. This motivates the ansatz

$$u = a\cos\theta + o(1), \qquad (1.1)$$ 01:eqn:01

with $a = a(\vec{X}, T)$, $\theta = \frac{1}{\epsilon}\Theta(\vec{X}, T)$, $\vec{X} = \epsilon\,\vec{x}$, and $T = \epsilon\,t$. Then $\theta \approx \vec{k}_0\cdot(\vec{x}-\vec{x}_0) - \omega_0\,(t-t_0) + \theta_0$ near any $(\vec{x}_0, t_0)$, where

$$\vec{k} = \partial_{\vec{X}}\,\Theta, \quad \omega = -\partial_T\,\Theta. \qquad (1.2)$$ 01:eqn:02

We have $\vec{k} = \partial_{\vec{x}}\,\theta$, $\omega = -\partial_t\,\theta$ as well.

Note 1.1 In a formal multiple-scales expansion (done elsewhere), one would write $u \sim A\,e^{i\,\theta} + \text{c.c.}$, and then expand $A = A_0(\vec{X}, T) + \epsilon\,A_1(\vec{X}, T) + \ldots$. ♣ 01:not:01

A direct consequence of (1.2) are the equations:

$$\vec{k}_T + \partial_{\vec{X}}\,\omega = 0 \quad \text{and} \quad \mathbf{curl}(\vec{k}) = 0, \qquad (1.3)$$ 01:eqn:03

where the first states that the *wave bumps are conserved,* while the second states that *wave crests do not end, suddenly, in space.* These equations are a *necessary condition for the existence of a phase* (sufficient over simply connected domains).

Using the expectation (justified below) that $\vec{k}$ and ω satisfy the dispersion relation, so that we can write $\omega = \Omega(\vec{k})$, (1.3) implies:

$$\vec{k}_T + (\vec{c}_g\cdot\partial_{\vec{X}})\,\vec{k} = 0, \qquad (1.4)$$ 01:eqn:04

where $\vec{c}_g = \partial_{\vec{k}}\,\Omega$ **is the group speed.** Further, it is easy to see † that, if $\mathbf{curl}(\vec{k}) = 0$ initially, then the time evolution by (1.4) preserves the property. Hence **we can replace (1.3) by (1.4),** provided **the initial conditions have zero curl.** Alternatively, we can solve the scalar equation

$$G\left(-\partial_T\,\Theta,\ \partial_{\vec{X}}\,\Theta\right) = 0. \qquad (1.5)$$ 01:eqn:05

Note: (1.4) states that **the wave frequency and wave number are carried by "pseudo-particles" moving along the paths**

$$\frac{\mathrm{d}\vec{X}}{\mathrm{d}T} = \vec{c}_g(\vec{k}). \qquad (1.6)$$ 01:eqn:06

† Let $k_{i,j} = \partial_{x_j}\,k_i$, $k_{i,jm} = \partial_{x_j}\partial_{x_m}\,k_i$, $\Omega_\ell = \partial_{k_\ell}\Omega = (c_g)_\ell$, and $\Omega_{\ell m} = \partial_{k_\ell}\partial_{k_m}\Omega$. Then, using the summation convention, (1.4) is $(k_i)_t + \Omega_\ell\,k_{i,\ell} = 0$. The derivative with respect to x_j yields: $(k_{i,j})_t + \Omega_\ell k_{i,\ell j} + \Omega_{\ell m}k_{i,\ell}k_{m,j} = 0$. This is: $K_t + D\,K + K\,W\,K = 0$, where K is the matrix with entries $k_{i,j}$, W the matrix with entries $\Omega_{\ell m}$, and $D = \Omega_\ell\,\partial_{x_\ell}$. Clearly K and K^T satisfy the same equation. Hence, if $K = K^T$ initially, it stays this way. ♣

Next we recall that **conservative physical systems** have a Lagrangian associated with them. Thus let

$$L = L(u, u_t, \nabla u, \ldots) \qquad (1.7)$$ 01:eqn:07

be the corresponding Lagrangian, where the dots indicate (possible) dependence on further space derivatives.

The governing equation follows from $\delta J = 0$, where

$$J[u] = \int_{t_1}^{t_2} \mathrm{d}t \int_{\mathcal{P}} L\,\mathrm{d}x_1\ldots\mathrm{d}x_d \qquad (1.8)$$ 01:eqn:08

is the action, $\boldsymbol{d}$ is the space dimension and $\boldsymbol{\mathcal{P}}$ is some region in space.

Thus (see §1.2, Euler-Lagrange equation)
$$\boldsymbol{L_u = \partial_t\, L_{u_t} + \partial_{x_j}\, L_{u_{x_j}} - \partial_{x_j}\partial_{x_i}\, L_{u_{x_j x_i}} + \ldots.} \quad (1.9)$$ 01:eqn:09

Notation.
- **We use the repeated index summation convention.**
- **∂_t, ∂_{x_j}, ..., denote "total" partial derivatives, while L_u, L_{u_t}, ..., denote derivatives with respect to the corresponding "slot" in L.**

Note 1.2 For simplicity we limit the treatment here to a scalar $u = u(\vec{x}, t)$, and a Lagrangian that depends only on u_t (i.e.: first or second order in time pde). Furthermore, we also assume linear waves; hence **the Lagrangian is homogeneous quadratic in u and its derivatives.** ♣ 01:not:02

Because we assume‡ that the solution has the form in (1.1), we can restrict the search for a stationary "point" of the action $\boldsymbol{J[u]}$ to functions $\boldsymbol{u}$ of the form in (1.1).

‡ This can be justified through a multiple scales expansion, which we will not show here, but can be found in [2].

Furthermore, because of the separation of scales between the "fast" dependence on $\boldsymbol{\theta}$ and the "slow" dependence on $\boldsymbol{(\vec{X}, T)}$, to leading order the integrations in (1.8) can be split into: First; integrate over $\boldsymbol{\theta}$ (i.e.: take the average over the periodic dependence). Second: integrate over the slow variables. This leads to the **averaged variational principle $\delta \bar{J} = 0$,**

where
$$\bar{J}[a, \Theta] = \int \mathrm{d}T \int \mathcal{L}\, \mathrm{d}X_1 \ldots \mathrm{d}X_d \quad (1.10)$$ 01:eqn:10

is the **wave action**, with the **average Lagrangian**

$\mathcal{L} = \mathcal{L}(a, \omega, \vec{k})$ given by
$$\mathcal{L} = \frac{1}{2\pi} \int_0^{2\pi} L(a\cos\theta, \omega\, a \sin\theta, -\vec{k}\, a \sin\theta, \ldots)\, \mathrm{d}\theta. \quad (1.11)$$ 01:eqn:11

Furthermore, note that $\mathcal{L}$ has the form
$$\mathcal{L} = c_d\, a^2\, G(\omega, \vec{k}), \quad (1.12)$$ 01:eqn:12

for some function G that must be the dispersion relation (see below), and some constant $c_d \neq 0$. This follows from note 1.2 ($\mathcal{L}$ is quadratic). The constant c_d follows from the fact that neither L, nor G, are unique (multiplying either by a nonzero constant has no effect). The relevant *Euler-Lagrange equations, corresponding to variations in* a *and* Θ, are 01:eqn:13

$$\mathcal{L}_a = 0 \;\; \text{(i.e.: } G(\omega, \vec{k}) = 0) \quad \text{and} \quad \partial_T\, \mathcal{L}_\omega - \partial_{X_j}\, \mathcal{L}_{k_j} = 0. \quad (1.13)$$

That G must be the dispersion relation follows by considering what the equation $\mathcal{L}_a = 0$ reduces to when a, ω, and $\vec{k}$, are constant. The second equation in (1.13) is a conservation equation, **the conservation of the wave action.**

G and the group speed are related by
$$G_\omega\, \vec{c}_g = -\partial_{\vec{k}}\, G, \quad (1.14)$$ 01:eqn:14

as follows by the chain rule used on $G = 0$. Using this we can write the **conservation of the wave action** in the form
$$\partial_T\, \mathcal{L}_\omega + \partial_{\vec{X}}\, (\vec{c}_g\, \mathcal{L}_\omega) = 0, \quad (1.15)$$ 01:eqn:15

i.e.: **the wave action flow speed is the group speed.**

Using this and (1.3–1.4) it is easy to see that† for any scalar valued function $\boldsymbol{f(\vec{k})}$. A particular case of this is the **conservation of energy**, since the energy density satisfies $\mathcal{E} = \omega\, \mathcal{L}_\omega$, as shown in (2.6).
$$\partial_T\, (f(\vec{k})\, a^2) + \partial_{\vec{X}}\, (\vec{c}_g\, f(\vec{k})\, a^2) = 0, \quad (1.16)$$ 01:eqn:16

† Write $f(\vec{k}) = g(\vec{k})\, G_\omega$, and use $\mathcal{L}_\omega = a^2\, G_\omega$ (O.K. to set $c_d = 1$).

Note 1.3 Why is $\mathcal{L}_\omega$ called the wave action density? The Lagrangian units are action per unit time and unit volume. Hence $\mathcal{L}_\omega$ is action per unit volume; i.e.: an "action density" associated with the wave. A better 01:not:03

way to motivate this is: for completely integrable classical Hamiltonian systems the "actions" in the action-angle variables are adiabatic invariants; while here $\mathcal{L}_\omega$ is an adiabatic invariant density. See note 1.4. ♣

Note 1.4 Generalization to a slowly varying Lagrangian. If L in (1.7) depends on the slow variables, $L = L(u,\, u_t,\, \nabla u,\, \ldots,\, \vec{X},\, T)$, the average Lagrangian approach still works. The changes that need to be introduced in the earlier results are as follows: 01:not:04

1. **(1.1–1.3) remain valid.** However, **since** $\omega = \Omega(\vec{k},\, \vec{X},\, T)$, 01:itm:01
   (1.4) changes to: $$\vec{k}_T + (\vec{c}_g \cdot \partial_{\vec{X}})\, \vec{k} = -\Omega_{\vec{X}}, \quad (1.17)$$ 01:eqn:17
   where $\vec{c}_g = \vec{c}_g(\vec{k},\, \vec{X},\, T) = \Omega_{\vec{k}}$.
2. **(1.5–1.6) change to:** $$G(-\partial_T\, \Theta,\, \partial_{\vec{X}}\, \Theta,\, \vec{X},\, T) = 0 \text{ and } \frac{\mathrm{d}\vec{X}}{\mathrm{d}T} = \vec{c}_g(\vec{k},\, \vec{X},\, T). \quad (1.18)$$ 01:eqn:18
3. **(1.7–1.11) remain valid.** 01:itm:03
4. **(1.12) changes to:** $$\mathcal{L} = c_d\, a^2 G(\omega,\, \vec{k},\, \vec{X},\, T). \quad (1.19)$$ 01:eqn:19
5. **(1.13–1.15) remain valid,** However 01:itm:05
   (1.16) changes to: $$\partial_T\, (g(\vec{k})\, \mathcal{L}_\omega) + \partial_{\vec{X}}\, (\vec{c}_g\, g(\vec{k})\, \mathcal{L}_\omega) = -g_{k_j}\, \Omega_{X_j}\, \mathcal{L}_\omega. \quad (1.20)$$ 01:eqn:20
   However, note that if we take
   $g = g(\vec{k},\, \vec{X},\, T)$, with $g_T + (c_g)_j\, g_{X_j} = g_{k_j}\, \Omega_{X_j}$ †
   (not clear that this can generally be done), then $$\partial_T\, (g(\vec{k})\, \mathcal{L}_\omega) + \partial_{\vec{X}}\, (\vec{c}_g\, g(\vec{k})\, \mathcal{L}_\omega) = 0. \quad (1.21)$$ 01:eqn:21
   † **If Ω is not T-dependent, $g = f(\Omega)$ satisfies this equation, leading to various conservation laws; e.g.: energy for $g = \Omega$** — see (2.6).

Since (1.15) remains valid, **the wave action is an adiabatic invariant** — see §3. However, (1.20–1.21) show that (in general) the energy is not conserved — it is when $L_T = 0$; see §2; in particular note 2.1. ♣

Note 1.5 Note that the average Lagrangian approach can be extended to nonlinear waves by considering slowly varying periodic traveling waves, and averaging over the periodic dependence. See [2]. ♣ 01:not:05

1.2 Variational derivative

A *functional* is a correspondence that assigns a number (real or complex) to every function in some set $\mathcal{S}$. Note that the set should have some notion of "size", i.e.: a norm. But here, for flexibility and simplicity, we will be somewhat informal. For the example in (1.22) a reasonable norm would be the maximum over $0 \le \ell \le n$ and t of $\|\vec{q}_\ell\|_2$.

For example, let the set be all the n-times continuously differentiable functions, $\vec{q} = \vec{q}(t)$, from some interval $t_1 \le t \le t_2$ to the γ-dimensional vector space $\mathbb{R}^\gamma$. Then define the functional J by $$J[\vec{q}] = \int_{t_1}^{t_2} L(\vec{q}_0,\, \ldots,\, \vec{q}_n,\, t)\, \mathrm{d}t, \quad (1.22)$$ 02:eqn:01
where L (the *Lagrangian*) is a scalar valued (smooth enough)† function, $\vec{q}_\ell = \mathcal{D}^\ell \vec{q}$, and $\mathcal{D} = \frac{\mathrm{d}}{\mathrm{d}t}$. In physics J is the *Action*.

† For what follows we will need $2n$-continuous derivatives.

By analogy with vector calculus, the derivative of a functional J (at a given function $\vec{q}$), $\delta J_{\vec{q}}$ (if it exists), is a *linear* functional with the property: $$J[\vec{q} + \delta\vec{q}] = J[\vec{q}] + \delta J_{\vec{q}}[\delta\vec{q}] + o(\delta\vec{q}). \quad (1.23)$$ 02:eqn:02

Example. Using integration by parts, it is straightforward (though cumbersome) to show that, for (1.22),

$$\delta J_{\vec{q}}[\delta\vec{q}] = \int_{t_1}^{t_2} \left(\sum_{\ell=0}^{n} (-1)^\ell \mathcal{D}^\ell \boldsymbol{L}_\ell \right) \cdot \delta\vec{q}_0 \, \mathrm{d}t \; + \left[\sum_{1\le j\le \ell\le n} (-1)^{j-1} \delta\vec{q}_{\ell-j} \cdot \mathcal{D}^{j-1} \boldsymbol{L}_\ell \right]_{t_1}^{t_2}, \tag{1.24}$$ 02:eqn:03

where $\boldsymbol{L}_\ell = \partial_{\vec{q}_\ell} \boldsymbol{L}$, evaluated at $(\vec{q}_0, \ldots, \vec{q}_n)$ — note that $\boldsymbol{L}_\ell$ is γ-vector valued.

Note 1.6 By analogy with the partial derivatives of vector calculus, we write 02:not:01

$$\left(\frac{\delta J}{\delta \vec{q}} \right)_{\vec{q}} = \sum_{\ell=0}^{n} (-1)^\ell \mathcal{D}^\ell \boldsymbol{L}_\ell. \tag{1.25}$$ 02:eqn:04

Note 1.7 Often, in Lagrangian mechanics, one assumes "boundary conditions" that eliminate the boundary terms in (1.24). For example: $\delta\vec{q}_j = \mathbf{0}$, for $\mathbf{0} \le j < n$, at both t_j. If not, the *Stationary Action Principle*, $\delta J = \mathbf{0}$, not only imposes the *Euler-Lagrange equations* but *natural boundary conditions* at the t_j. 02:not:02

$$\sum_{\ell=0}^{n} (-1)^\ell \mathcal{D}^\ell \boldsymbol{L}_\ell = \mathbf{0}, \tag{1.26}$$ 02:eqn:05

Note 1.8 Of course, all of this can be extended to functions of several variables (where the resulting Euler-Lagrange equations are pde instead of ode), but we will not do this here — this sub-section is intended as a short example/refresher only. For *dispersive waves* the analog of (1.22) typically involves integration over a *cylindrical region* $\mathcal{R} \times (t_1, t_2)$ in space-time, where $\mathcal{R}$ is the region in space where the waves occur. Then $\boldsymbol{L}$ depends, in addition, on space derivatives of the solutions. ♣ 02:not:03

1.3 Slowly varying versus high frequency

Here we point out that the difference between the *slowly varying wave theory* presented earlier, and the *high frequency approximation* which is often found in optics, ultrasound, etc., is merely **a question of which space and time scales are used produce a-dimensional variables.** *Both correspond to situations with nearly plane waves over the wave scales, with the wave parameters varying over many wavelengths/wave periods.* If a-dimensional variables are obtained by using typical wavelength/wave period scales, then a slowly varying waves "picture" results. On the other hand, if the scales over which the wave parameters change are used, then a high frequency viewpoint results.

In terms of the theory developed earlier in this section, the *switch to a high frequency approximation* merely involves changing (1.1) to $\boldsymbol{u} = \boldsymbol{a}\cos\boldsymbol{\theta}$, with $\boldsymbol{a} = \boldsymbol{a}(\vec{x}, t)$ and $\boldsymbol{\theta} = \frac{1}{\epsilon}\Theta(\vec{x}, t)$. Generally, map: $\vec{x} \to \vec{x}/\epsilon$, $t \to t/\epsilon$, $\vec{X} \to \vec{x}$, and $T \to t$.

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2 Noether's Theorem

Relationship between the conservation of the wave action and the energy.

Noether's Theorem (E. Noether, 1915) states that: *Assume a system characterized by a Lagrangian, $\boldsymbol{L}$. Then: for every differentiable symmetry under which $\boldsymbol{L}$ is invariant there is an associated conservation law* (for examples illustrating the theorem's proof, see §2.1). In particular:

6. Invariance under translation in space corresponds to the conservation of momentum. itm:01

7. Invariance under rotation in space corresponds to the conservation of angular momentum. itm:02
8. Invariance under translation in time corresponds to the conservation of energy. itm:03

Here the focus is on energy conservation. We work by examples rather than "most general theory".

Notation.
- **We use the repeated index summation convention.**
- **∂_t, ∂_{x_ℓ}, ..., denote "total" partial derivatives, while L_t, $\mathcal{L}_T$, ..., denote derivatives with respect to the corresponding "slot" in L, $\mathcal{L}$, ...**

Consider a linear wave governed by a Lagrangian of the form $$L = L(u,\, u_t,\, \nabla u). \tag{2.1}$$ eqn:01

Since the wave is linear, **the Lagrangian is homogeneous quadratic in u and its derivatives.** Since L does not depend on time, **conservation of energy** applies † $$\partial_t(u_t\, L_{u_t} - L) + \partial_{x_\ell}(u_t\, L_{u_{x_\ell}}) = 0. \tag{2.2}$$ eqn:02

† See note 2.3.

Let us now use the slowly varying wave ansatz (1.1) on this equation. We have:

$$u_t\, L_{u_t} - L = \omega\, a\, \sin\theta\, L_{u_t}\ (a\, \cos\theta,\, \omega\, a\, \sin\theta,\, -\vec{k}\, a\, \sin\theta) + o(1), \text{ and}$$
$$u_t\, L_{u_{x_\ell}} \quad = \omega\, a\, \sin\theta\, L_{u_{x_\ell}}(a\, \cos\theta,\, \omega\, a\, \sin\theta,\, -\vec{k}\, a\, \sin\theta) + o(1).$$

Thus, the (leading order) average energy density is $$\mathcal{E} = \omega\, \mathcal{L}_\omega - \mathcal{L}, \tag{2.3}$$ eqn:03

where $\mathcal{L} = \mathcal{L}(\omega,\, \vec{k})$ is the average Lagrangian. Similarly, the (leading order) average energy flux components are $$\mathcal{F}_\ell = -\omega\, \mathcal{L}_{k_\ell}. \tag{2.4}$$ eqn:04

Note that **$\lambda\, \mathcal{E}$ is the energy in each wave.**

Thus, averaging of (2.2) leads ‡ to the **conservation of the wave energy** law $$\partial_T\, (\omega\, \mathcal{L}_\omega - \mathcal{L}) - \partial_{X_\ell}\, (\omega\, \mathcal{L}_{k_\ell}) = 0. \tag{2.5}$$ eqn:05

Not surprisingly, this is **exactly the same conservation law that follows from the fact that the average Lagrangian is time independent.**

‡ Note that $\partial_t = -\omega\, \partial_\theta + \epsilon\, \partial_T$ and $\partial_{x_\ell} = k_\ell\, \partial_\theta + \epsilon\, \partial_{X_\ell}$. Thus averaging eliminates the derivatives with respect to θ and leaves a conservation law in terms of T and $\vec{X}$.

Note 2.1 For linear waves $\mathcal{L} = a^2\, G(\omega,\, \vec{k})$, and $G(\omega,\, \vec{k}) = 0$ (Dispersion relation). Hence $\mathcal{L} = 0$, and (2.6) yields $$\mathcal{L}_\omega = \mathcal{E}/\omega. \tag{2.6}$$ not:02 eqn:05

This relationship between the wave energy density and the wave action density applies for linear waves only. Nevertheless, it "explains" how the wave action remains conserved even when the energy is not conserved (see below) — i.e.: **the wave action is an adiabatic invariant** (see § 3). The "how" is that when the Lagrangian is time dependent, both the wave energy density and the wave frequency change following the group speed, but they do so (at least for linear waves) in such a way that the ratio in (2.6) is conserved. ♣

These facts are not "intuitively obvious", and proving them by direct frontal attack (e.g.: formal perturbation expansions) is anything but trivial. Yet they follow easily from the powerful variational calculus machinery.

If (2.1) is changed so that $$L = L(u,\, u_t,\, \nabla u,\, \vec{X},\, T), \tag{2.7}$$ eqn:07

the average Lagrangian machinery still works. However, the **results above must be modified as follows:**

9. **If $L_T = 0$**, no changes are needed — other than the fact that now $\mathcal{L} = a^2\, G(\omega,\, \vec{k},\, \vec{X})$. itm:04
10. **If $L_T \neq 0$**, then (2.2) must be changed to $$\partial_t(u_t\, L_{u_t} - L) + \partial_{x_\ell}(u_t\, L_{u_{x_\ell}}) = -\epsilon\, L_T, \tag{2.8}$$ eqn:08

while (2.5) must be changed to $$\partial_T\,(\omega\,\mathcal{L}_\omega-\mathcal{L})-\partial_{X_\ell}\,(\omega\,\mathcal{L}_{k_\ell})=-\mathcal{L}_T. \tag{2.9}$$ eqn:09

Hence the **conservation of wave energy fails,**
but **the wave action is still conserved:** $$\partial_T\;\mathcal{L}_\omega-\partial_{X_\ell}\,\mathcal{L}_{k_\ell}=0. \tag{2.10}$$ eqn:10

2.1 Example proofs of Noether's theorem

Consider the example in (1.22). Assume that $\boldsymbol{L}$ is time independent, so it is is invariant under the symmetry $\boldsymbol{t\to t+s}$ and $\boldsymbol{\vec q\to\vec q}$. Let us see what this means if $\boldsymbol{\vec q}$ satisfies the Euler-Lagrange equations (1.26). Let $\boldsymbol{\vec p=\vec q(t+\mathrm{d}s)}$. Then $\boldsymbol{\vec p=\vec q+\delta\vec q}$, with $\boldsymbol{\delta\vec q=\vec q_1\mathrm{d}s}$. Furthermore:

$$0=\int_{t_1-\mathrm{d}s}^{t_2-\mathrm{d}s}L\{\vec p\}\,\mathrm{d}t-J[\vec q]=[-L\{\vec q\}]_{t_1}^{t_2}\,\mathrm{d}s+J[\vec p]-J[\vec q]+o(\mathrm{d}s),$$

where **we use the notation in (1.22–1.26), and $L\{\vec p\}=L(\vec p_0,\,\ldots,\,\vec p_n)$.** Hence upon using (1.24), the fact that $\boldsymbol{\vec q}$ satisfies the Euler-Lagrange equations, and $\boldsymbol{\delta\vec q=\vec q_1\mathrm{d}s}$.

$$0=\left[\sum_{1\le j\le\ell\le n}(-1)^{j-1}\vec q_{\ell-j+1}\cdot\mathcal{D}^{j-1}\,L_\ell-L\right]_{t_1}^{t_2}\mathrm{d}s+o(\mathrm{d}s).$$

Note that here $\boldsymbol{L}$ and its derivatives are evaluated at $\boldsymbol{\vec q}$ and its derivatives. It follows that NP:eqn:01

$$E=\sum_{1\le j\le\ell\le n}(-1)^{j-1}\vec q_{\ell-j+1}\cdot\mathcal{D}^{j-1}\,L_\ell-L=\text{"energy"}\quad\text{is conserved.} \tag{2.11}$$

Note that the "physical" energy would be αE, for some constant α; at this level of abstraction we cannot determine α.

† **Notation:** $L\{\vec p\}=L(\vec p_0,\,\ldots,\,\vec p_n)$.

Examples.

11. For $n=1$, $E=\vec q_1\cdot L_1-L$. NP:itm:01
12. For $n=2$, $E=-\vec q_1\cdot\mathcal{D}L_2+\vec q_2\cdot L_2+\vec q_1\cdot L_1-L$. NP:itm:02

Note 2.2 The proof for other symmetries is similar. **The "boundary terms" in (1.24) are always the key.** While the symmetry preserves $\boldsymbol{L}$, it changes the boundary terms. Hence, since J on the (infinitesimally) transformed variables must equal J on the original ones, the difference reduces to contributions from the boundaries — which must add to zero, yielding the conservation law. *For pde variational principles,* boundary terms in space arise as well: then *the time boundary terms provide the conserved densities and the space boundary terms provide the fluxes.* ♣ NP:not:01

2.1.1 Examples for pde

Consider the functional $$J[u]=\int_{t_1}^{t_2}\mathrm{d}t\int_{x_1}^{x_2}\mathrm{d}x\;L(u,\,u_t,\,u_x), \tag{2.12}$$ NP:pde:eqn:01
defined for scalar functions $\boldsymbol{u=u(x,\,t)}$. Then NP:pde:eqn:02

$$\delta J_u[\delta u]=\int_{t_1}^{t_2}\mathrm{d}t\int_{x_1}^{x_2}\mathrm{d}x\,(L_0-\partial_tL_1-\partial_xL_2)\;\delta u+\left[\int_{x_1}^{x_2}\mathrm{d}x\,L_1\,\delta u\right]_{t_1}^{t_2}+\left[\int_{t_1}^{t_2}\mathrm{d}t\,L_2\,\delta u\right]_{x_1}^{x_2}, \tag{2.13}$$

where $\boldsymbol{L_0=\partial_uL}$, $\boldsymbol{L_1=\partial_{u_t}L}$, $\boldsymbol{L_2=\partial_{u_x}L}$, and the $\boldsymbol{L_j}$ are evaluated at $\boldsymbol{(u,\,u_t,\,u_x)}$.
The meaning of $\boldsymbol{L_j}$ here is similar to the one in §1.2 and §2.1; $\boldsymbol{L_j}$ is the partial derivative with respect to the slot $\boldsymbol{j+1}$.
The corresponding Euler-Lagrange equation is then $$L_0-\partial_tL_1-\partial_xL_2=0. \tag{2.14}$$ NP:pde:eqn:03

Assume now that u **solves this equation.**

Conservation of energy. Let $v = u(t + \mathrm{d}s,\, x) = u + \delta u$, where $\delta u = u_t\, \mathrm{d}s$. Then

$$\begin{aligned} 0 &= \int_{t_1-\mathrm{d}s}^{t_2-\mathrm{d}s} \mathrm{d}t \int_{x_1}^{x_2} \mathrm{d}x\, L\{v\} - J[u] = \left[-\int_{x_1}^{x_2} \mathrm{d}x\, L\right]_{t_1}^{t_2} \mathrm{d}s + J[v] - J[u] + o(\mathrm{d}s) \\ &= \left[\int_{x_1}^{x_2} \mathrm{d}x\, (u_t\, L_1 - L)\right]_{t_1}^{t_2} \mathrm{d}s + \left[\int_{t_1}^{t_2} \mathrm{d}t\, u_t\, L_2\right]_{x_1}^{x_2} \mathrm{d}s + o(\mathrm{d}s) \\ &= \int_{t_1}^{t_2} \mathrm{d}t \int_{x_1}^{x_2} \mathrm{d}x \left(\partial_t(u_t\, L_1 - L) + \partial_x(u_t\, L_2)\right) \mathrm{d}s + o(\mathrm{d}s), \end{aligned}$$

where (unless indicated) L and L_j are evaluated at $(u,\, u_t,\, u_x)$. Since the t_j and x_j are arbitrary,

$$\partial_t(u_t\, L_{u_t} - L) + \partial_x(u_t\, L_{u_x}) = 0. \quad (2.15)$$ NP:pde:eqn:04

This is the **equation for the conservation of energy**, with **energy density** $u_t\, L_{u_t} - L$ and **energy flux** $u_t\, L_{u_x}$.

Note 2.3 It is easy to generalize this to d dimensions, with $L = L(u,\, u_t,\, \nabla u)$. Then the energy density is still $u_t\, L_{u_t} - L$, while the d components of the flux are $u_t\, L_{u_{x_\ell}}$. NP:pde:not:01

Furthermore: notice that, **if L depends on higher order derivatives of u** (beyond ∇u), **the energy flux changes, but the energy density does not.**‡ ♣

‡ Reason: adding higher order space derivatives of u to the arguments in L introduces extra space boundary terms in (2.13), but does not change the time boundary terms. Thus, in the argument above, the same energy density results.

Conservation of momentum.

The same process† leading to (2.15) yields

$$\partial_t(u_x\, L_{u_t}) + \partial_x(u_x\, L_{u_x} - L) = 0. \quad (2.16)$$ NP:pde:eqn:05

Note 2.4 It is easy to generalize this to d dimensions, with $L = L(u,\, u_t,\, \nabla u)$. For every direction $1 \le j \le d$, the momentum density is $u_{x_j} L_{u_t}$, while the d components of the corresponding flux are $u_{x_j}\, L_{u_{x_\ell}} - \delta_{j\ell} L$. NP:pde:not:02 ♣

† Simply switch the roles of x and t in the calculation!

Note 2.5 For **conservation of angular momentum** it would have to be $L = L(u,\, u_t,\, \|\nabla u\|_2)$. We will not consider this example since it is not relevant to these notes. NP:pde:not:03 ♣

Note 2.6 **Generalizations to other symmetries, cases where L involves higher order derivatives of $u = u(\vec{x},\, t)$, or situations where $\vec{u}$ is vector valued,** are technically more complicated (and require the use of good notation), but follow the same basic process. NP:pde:not:04 ♣

3 Adiabatic invariants and canonical transformations

3.1 Adiabatic invariants

Consider a slowly varying smooth Hamiltonian system in 2 dimensions, with Hamiltonian $H = H(\vec{z},\, \tau)$, where $\vec{z} = (\vec{q},\ \vec{p})$, $\tau = \epsilon\, t$, and $0 < \epsilon \ll 1$. Assume that, **for each frozen τ** (at least in some open τ-interval), **the Hamiltonian has a local (strict) minimum** (or a maximum) **that may change, but it is**

not lost, at some $\vec{z} = \vec{z}_m$. Thus, near $\vec{z} = \vec{z}_m$, **for each frozen τ, the system has a smooth one parameter family of periodic solutions.** Specifically:

$\vec{z} = \vec{Z}(\theta, E, \tau)$, with $\theta = \omega(E, \tau)\, t$, satisfying $\omega\, \partial_\theta Q = \partial_p H$ and $\omega\, \partial_\theta P = -\partial_q H$, (3.1) AI:eqn:01

where $\vec{Z}$ is 2π-periodic in θ, and E is the energy of the solution — i.e.: $E = H(\vec{Z}, \tau)$.

As an illustration of this, see example 3.1, with $V = V(q, \tau)$. Then Q is given (implicitly) by the last formula in (3.43), ω by the first formula, and $P = \sqrt{2\,(E - V(Q, \tau)}$.

Next **assume a "τ not frozen solution"** $\vec{z} = \vec{Z}(\theta, E, \tau) + O(\epsilon)$, (3.2) AI:eqn:02

where $E = E(\tau)$ and $\theta = \frac{1}{\epsilon}\Theta(\tau)$.

Next we proceed as in § 1.1. We substitute (3.2) into the Lagrangian $L = p\,\dot{q} - H$ for the Hamiltonian system, and compute the average Lagrangian $\mathcal{L}$ — i.e.: average over one period in θ. This yields $\mathcal{L} = \Omega\, J(E, \tau) - E$, (3.3) AI:eqn:03

where $\Omega = \mathrm{d}\Theta/\mathrm{d}\tau$, and J is the action variable (see § 3.2.2 and 3.2.4)

$$J = \frac{1}{2\,\pi}\int_0^{2\,\pi} (P\,\partial_\theta Q)\,\mathrm{d}\theta, \quad (3.4) \text{ AI:eqn:04}$$

The associated Euler-Lagrange are then $\dfrac{\mathrm{d}}{\mathrm{d}\tau}\partial_\Omega \mathcal{L} = 0$, and $\partial_E \mathcal{L} = 0$. That is[†]

$$\frac{\mathrm{d}J}{\mathrm{d}\tau} = 0 \text{ and } \Omega = \omega(E, \tau). \quad (3.5) \text{ AI:eqn:05}$$

In particular: ***J* is an adiabatic invariant:** it does not change when the Hamiltonian changes slowly. Notice: actually, it changes, but on a very slow time scale, at least $O(\epsilon^2)$ — see notes 3.6 and 3.7.

† **Proof of (3.5).** The first equation is obvious. The second follows because $\partial_E J = 1/\omega$. [A].

To prove [A], note that $0 = \partial_E\,(E - H) = 1 - \partial_q H\,\partial_E Q - \partial_p H\,\partial_E P$

yields, upon use of (3.1), $\partial_\theta Q\,\partial_E P - \partial_\theta P\,\partial_E Q = 1/\omega$. [B]

On the other hand $\partial_E(P\,\partial_\theta Q) = \partial_E P\,\partial_\theta Q - \partial_\theta P\,\partial_E Q + \partial_\theta(P\,\partial_E Q)$. [C]

Substituting [C] into ∂_E of (3.4), and using [B], yields [A]. **QED**

Remark 3.1 Why is (3.2) valid/justified? Let $\vec{z}_0 = \vec{z}(t_0)$ be the value of a solution at some time t_0, and let $\vec{\zeta}_0(t)$ be the solution of the system frozen at $\tau = \epsilon\, t_0$ satisfying $\vec{\zeta}_0(t_0) = \vec{z}_0$. Then, for a not-too-large time interval $t_0 - \Delta t < t < t_0 + \Delta t$, $\vec{z}$ and $\vec{\zeta}$ will be within $O(\epsilon)$ of each other. Furthermore (3.1) includes **all** the solutions close to $\vec{z}_m$, hence (provided $\vec{z}_0$ is close to $\vec{z}_m$), $\vec{\zeta}_0$ corresponds to (3.1) for some value $E = E_0$ and choice of phase $\theta = \theta_0$ at $t = t_0$. This argument shows that the "τ not frozen solutions" can be locally approximated by (3.1), but the solution used must change in time. **This is exactly what (3.2) implements.** AI:rm:01

The same argument applies for the 2d dimension calculation below, and justifies (FILL IN). It also explains why complete integrability, with quasi-periodic solutions, is needed: the solutions used, i.e.: (FILL IN), must fill an open region in phase space, else there is no way to ensure that the "τ not frozen solution" can, at each time, be approximated by a solution in **(FILL IN)**. ♣

Remark 3.2 Annular region extension. Note that families of periodic solutions like the one in (3.1) exist not only in the neighborhood of a local minimum or maximum of H, but can arise in annular regions and other situations — see example 3.1-items **(e-f)**. Regardless of particular details, the arguments in remark 3.1 show that **if a smooth one parameter family of periodic solutions completely fills in an open region, then the approach in (3.2–3.5) is valid.** ♣ AI:rm:02

Remark 3.3 Accuracy may be higher than indicated by (3.2). The argument in remark 3.1 suggests AI:rm:03

that the error in the approximation is $O(\epsilon)$, as shown in (3.2). However, this is the error to be expected if one keeps τ frozen and changes t by $O(1)$. **Might not the error be smaller because the adaptation of E and θ is continuous?** *I do not know the answer in general*, but **the example of the harmonic oscillator, $H = \frac{1}{2}\,(p^2 + \omega^2(\tau)\,q^2)$ indicates that this may be the case.** Then $\ddot{q} + \omega^2\,q = 0$ ($p = \dot{q}$), and (3.2–3.5) is the leading order in the WKBJ expansion $q \sim A\cos\theta$, where $\theta = \frac{1}{\epsilon}\int^\tau \omega(s)\,ds$, $A = a_0(\tau) + \epsilon\,a_1(\tau) + \ldots$, and the adiabatic invariant surfaces in the fact that $\omega\,a_0^2$ is constant. This expansion is valid to all orders in ϵ, and the form $A\cos\theta$ is exactly that of the periodic solution. The **difference with (3.2–3.5) is that higher order corrections to E need to be included.** Surprisingly, **it is only the adiabatic invariant $\omega\,A^2$ that fails to be invariant to order higher than ϵ^2.** ♣

Can we generalize (3.2–3.5) to 2d dimension, $d > 1$, Hamiltonians?
The answer is **yes, provided the Hamiltonian is completely integrable.**‡ (3.6) AI:eqn:06

‡ Such systems are rare. They require the existence of d integral of motion (not just H, but $d-1$ more). They can be transformed (via a canonical transformation) into action-angle variables $(\vec{\theta}, \vec{J})$, where H depends on $\vec{J}$ only.

Remark 3.4 Hamiltonian systems are too big a class. The fact that to extend the adiabatic invariant theory we need to restrict the Hamiltonians used might bother you, but it should not be a surprise. *Any theory that applies to all Hamiltonian systems, applies to all continuous dynamical systems as well*, which is clearly too much to expect in general. The reason for this is: AI:rm:04

d-dim continuous dynamical systems can be embedded into 2d-dim Hamiltonian systems.

Proof. Given a d-dimension ode $\frac{\mathrm{d}}{\mathrm{d}t}\vec{q} = \vec{F}(\vec{q},\,t)$, consider the Hamiltonian $H = \vec{p}^{\,T}\vec{F}(\vec{q},\,t)$. This yields the equations: $\frac{\mathrm{d}}{\mathrm{d}t}\vec{q} = H_{\vec{p}} = \vec{F}(\vec{q},\,t)$, and $\frac{\mathrm{d}}{\mathrm{d}t}\vec{p} = -H_{\vec{q}} = J(\vec{q},\,t)\,\vec{p}$, where J is the matrix with entries $J_{nm} = -\partial_{q_n}F_m$. **QED**
Notice that **for infinite dimensions this breaks down.** For example, if $q_t = q_{xx}$, then $H = \int p\,q_{xx}\,\mathrm{d}x$ leads to $p_t = -p_{xx}$, which is ill-posed (hence H is not a valid Hamiltonian). ♣

To do: adiabatic invariants for completely integrable Hamiltonians.

3.2 Canonical transformations and action-angle variables

Here we review canonical transformations for Hamiltonian systems, and give some simple examples.

3.2.1 Canonical transformations and generating functions

Here we only consider the case of Hamiltonians in 2D (one q and one p). However, **the generalization to the case where $\vec{q}$ and $\vec{p}$ are vectors is straightforward** — in most cases it amounts to replacing products like $p\,q$ by $\vec{p}\cdot\vec{q}$.

A transformation between Hamiltonian systems $(q,\,p,\,H) \longleftrightarrow (Q,\,P,\,K)$ is **canonical** if its associated Lagrangians are equivalent. Hence the Hamiltonian form is preserved. That is

$$\lambda\,[p\,\dot{q} - H(p,\,q,\,t)] = P\,\dot{Q} - K(P,\,Q,\,t) + \frac{\mathrm{d}\mathcal{G}}{\mathrm{d}t}, \qquad (3.7)$$ GeF:eqn:01

where λ is a nonzero constant. Note: **a sequence of canonical transformations is canonical.** (3.8) GeF:eqn:02

Coordinate reversal $(q,\,p,\,H) \longleftrightarrow (p,\,q,\,-H)$ **is canonical.** (3.9) GeF:eqn:03
It corresponds to $\lambda = -1$ and $\mathcal{G} = -p\,q$.

Coordinate scaling ($\alpha \neq 0$ constant) $(q,\,p,\,H) \longleftrightarrow (\alpha\,q,\,\alpha\,p,\,\alpha^2 H)$ **is canonical.** (3.10) GeF:eqn:04

It corresponds to $\lambda = \alpha^2$ and $\mathcal{G} = 0$.

From (3.8–3.10) it follows that **there is no loss of generality in the restriction $\lambda = 1$.** (3.11) GeF:eqn:05

Note 3.1 **Generating functions.** (3.7) can be written as $\mathrm{d}\mathcal{G} = p\,\mathrm{d}q - P\,\mathrm{d}Q + (K - H)\,\mathrm{d}t$. Using that $p\,\mathrm{d}q = -q\,\mathrm{d}p + \mathrm{d}(p\,q)$ and $P\,\mathrm{d}Q = -Q\,\mathrm{d}P + \mathrm{d}(P\,Q)$, we can write $\mathcal{G} = G + \sigma_1\,p\,q - \sigma_2\,P\,Q$ in such a way that $\mathrm{d}G$ involves only $\mathrm{d}z$, $\mathrm{d}Z$, and $\mathrm{d}t$. GeF:not:01

Here: (i) $\sigma_j = 0$ or $\sigma_j = 1$, and (ii) $z = q$ or $z = p$, and $Z = Q$ or $Z = P$.

Then, taking $G = G(z, Z, t)$, and assuming that z and Z are independent, **(3.7) completely determines the transformation.** G is called the **generating function.**

Example: for the case in (3.12), $G_p\,\mathrm{d}p + G_Q\,\mathrm{d}Q + G_t\,\mathrm{d}t = -q\,\mathrm{d}p - P\,\mathrm{d}Q + (K - H)\,\mathrm{d}t$. ♣

Case $\mathcal{G} = G(p, Q, t) + p\,q$. $q = -G_p$, $P = -G_Q$, and $K = H + G_t$. (3.12) GeF:eqn:06

Case $\mathcal{G} = G(p, P, t) + p\,q - P\,Q$. $q = -G_p$, $Q = +G_P$, and $K = H + G_t$. (3.13) GeF:eqn:07

Case $\mathcal{G} = G(q, Q, t)$. $p = +G_q$, $P = -G_Q$, and $K = H + G_t$. (3.14) GeF:eqn:08

Case $\mathcal{G} = G(q, P, t) - P\,Q$. $p = +G_q$, $Q = +G_P$, and $K = H + G_t$. (3.15) GeF:eqn:09

Note 3.2 **At least locally there is (almost) always a generating function when $\lambda = 1$.** Given a canonical transformation, (3.7) will apply for some $\mathcal{G}$. Then: (i) Assume (p, Q) — or (p, P), or (q, Q), or (q, P) — is a local coordinate system in phase space. (ii) Write $\mathcal{G} = \mathcal{G}(p, Q)$ — or $\mathcal{G} = \mathcal{G}(p, P)$, or ... (iii) Define G using the corresponding case in (3.12–3.15). ♣ GeF:not:02

Note 3.3 **Generalization to systems: $\vec{q}$ and $\vec{p}$ are d-vectors.** Then the **generating function has the form** $G = G(\vec{s}, \vec{S}, t)$, where (for any $1 \le j \le d$) either $s_j = q_j$ or $s_j = p_j$, while $S_j = Q_j$ or $S_j = P_j$. Then the corresponding formulas in (3.12–3.15) are applied to each component, with: $\mathcal{G} = G + \sum \eta_j$, where (i) $\eta_j = p_j\,q_j$ if $s_j = p_j$ and $S_j = Q_j$; (ii) $\eta_j = p_j\,q_j - P_j\,Q_j$ if $s_j = p_j$ and $S_j = P_j$; etc. ♣ GeF:not:03

Note 3.4 **The cross derivative of G must not vanish, e.g.: for (3.15) $G_{q\,P} \neq 0$. This is so that the transformation is, at least locally, invertible.** In the vector case this generalizes to $\det(G_{s_n, S_n}) \neq 0$. GeF:not:04

Consider the (3.15) case. To have $p = G_q(q, P, t)$ properly define (at least locally) $P = P(p, q, t)$, we need $G_{qP} \neq 0$. Then $P_p = 1/G_{qP} = \mu$, $P_q = -\mu\,G_{qq}$, $Q_p = \mu\,G_{PP}$, $Q_q = \mu^{-1} - \mu\,G_{qq}G_{PP}$, and (finally) Jacobian $= P_p\,Q_q - P_q\,Q_p = 1$. The other cases are similar **(left to the reader to verify).** ♣

Note 3.5 **Time scale changes.** Obviously **Hamiltonian systems** are also **invariant under time scale changes**, even though **these are not canonical transformations.** GeF:not:05

For example, $(q, p, H, t) \to (Q, P, K, \tilde{t})$ (3.16) GeF:eqn:10

(with $t = \alpha\,\tilde{t}$, $Q = q$, $P = \beta\,p$, and $K = \gamma\,H$), preserves the Hamiltonian form provided that (α, β, γ) are nonzero constants satisfying $\alpha\,\beta = \gamma$. It is also desirable to have $\alpha > 0$ to preserve the "arrow of time", even though Hamiltonians are time reversible.†

† The system is invariant under flipping the sign of time and p. ♣

3.2.2 Harmonic oscillator in action-angle variables

Here we show the reduction of the harmonic oscillator to action-angle variables. **The case with a variable frequency is included,** to illustrate how the introduction of time dependence affects the transformation.

This also provides an **alternative way to look at adiabatic invariants.**

Consider the Hamiltonian $$H = \frac{1}{2}\,(p^2 + \omega^2\, q^2), \quad (3.17)$$ HAA:eqn:01

where $\omega = \omega(t) > 0$. The Lagrangian is $p\,\dot q - H$, and the equations are $\dot p = -H_q = -\omega^2\, q$ and $\dot q = H_p = p$.

Let now $$p = \sqrt{2\,\omega\, J}\,\cos\theta \;\text{ and }\; q = \sqrt{2\,J/\omega}\,\sin\theta, \quad (3.18)$$ HAA:eqn:02

with $J > 0$ and $K = H + (\dot\omega/\omega)\, J\, \cos\theta\, \sin\theta$.

Of course, when ω is constant, (3.18) solves the equations with: J constant, $\theta = \omega\, t + \theta_0$, and θ_0 constant

Then **(3.18) is canonical, with J the new momentum.** In the new variables $$H = \omega\, J. \quad (3.19)$$ HAA:eqn:03

Proof. It is easy to see that $(p\,\dot q - H) - (J\,\dot\theta - K) = \frac{\mathrm{d}}{\mathrm{d}t}(J\,\cos\theta\,\sin\theta)$.

Generating function. This corresponds to (3.12), $$G = -\frac{1}{2\,\omega}\,p^2\,\frac{\sin\theta}{\cos\theta}. \quad (3.20)$$ HAA:eqn:04

It is straightforward to check that $\mathcal{G} = G + p\,q = J\,\cos\theta\,\sin\theta$, $G_p = -q$, $G_\theta = -J$, and $G_t = (\dot\omega/\omega)\, J\,\cos\theta\,\sin\theta$.

The transformed equations are $$\dot\theta = +K_J = \omega + (\dot\omega/\omega)\,\cos\theta\,\sin\theta, \quad (3.21)$$ HAA:eqn:05

and $$\dot J = -K_\theta = (\dot\omega/\omega)\, J\,(\sin^2\theta - \cos^2\theta). \quad (3.22)$$ HAA:eqn:06

Finally (see remark 3.5) $$J = -\frac{1}{2\,\pi}\oint p\,\mathrm{d}q = \frac{1}{2\,\pi}\,A, \quad (3.23)$$ HAA:eqn:07

where A is the area enclosed by the (frozen time) periodic orbits — these are ellipses, with minor and major radius $\sqrt{2\,J\,\omega}$ and $\sqrt{2\,J/\omega}$; hence $A = \pi\, r_{min}\, r_{max}$.

Of course, **when $\dot\omega = 0$ the system is reduced to action-angle variables.**

HAA:rem:01

Remark 3.5 On signs and circulations. Here we have "reversed" the standard polar coordinates in the $(q,\,p)$-plane — using $\sin\theta$ for q and $\cos\theta$ for p. This is to avoid having to use $\omega < 0$ and $J < 0$, as the generic analysis in item 20 in § 3.2.4 indicates. This also triggers the negative sign in (3.23), since the orbit is traversed clockwise: $\oint p\,\mathrm{d}q = -A$. ♣

HAA:rem:02

Remark 3.6 Geometrical interpretation of θ (this is generalized in § 3.2.4). Define:
(a) $\mathcal{O}(E)$ = orbit corresponding to some energy $E = \frac{1}{2}\,(p^2 + \omega^2\, q^2) > 0$.
(b) $\mathcal{R}(E)$ = annular region contained between $\mathcal{O}(E)$ and $\mathcal{O}(E + \mathrm{d}E)$, with total area $A_{\mathcal{R}} = \frac{2\,\pi}{\omega}\,\mathrm{d}E$.
(c) A_ϕ = area fraction of $\mathcal{R}(E)$, measured counterclockwise, between $\theta = 0$ and $\theta = \phi$.

Then $\theta = 2\,\pi\, A_\theta/A_{\mathcal{R}}$.

Proof. (I) $\mathcal{O}(E)$ is an ellipse, with minor/major radius $\sqrt{2\,J\,\omega}$ and $\sqrt{2\,J/\omega}$. It encloses the area $A_{\mathcal{O}} = 2\,\pi\,J = 2\,\pi\,E/\omega$, which yields $A_{\mathcal{R}}$ above. (II) The orbit is $\vec z = (q,\,p) = \frac{\sqrt{2\,E}}{\omega}\,(\sin\theta,\,\omega\,\cos\theta)$, with external unit normal $\hat n = \frac{1}{\mu}(\omega\,\sin\theta,\,\cos\theta)$, where $\mu = \sqrt{\omega^2\,\sin^2\theta + \cos^2\theta}$. It follows that the distance between $\mathcal{O}(E)$ and $\mathcal{O}(E + \mathrm{d}E)$ is $\mathrm{d}\delta = \hat n\cdot\frac{\mathrm{d}\vec z}{\mathrm{d}E}\,\mathrm{d}E = \frac{1}{\mu\,\sqrt{2\,E}}\,\mathrm{d}E$, while the arc-length is $\mathrm{d}s = \frac{\sqrt{2\,E}}{\omega}\,\mu\,\mathrm{d}\theta$. Hence, upon integration in θ, $A_\phi = \int_0^\phi \mathrm{d}\delta\,\mathrm{d}s = \frac{1}{\omega}\,\phi\,\mathrm{d}E$, which yields $\theta = 2\,\pi\,A_\theta/A_{\mathcal{R}}$. ♣

Adiabatic invariants and action-angle variables.

Note 3.6 Action variable behavior for slowly varying frequency: adiabatic invariant. HAA:not:01

Assume that $\omega = \omega(\tau)$, where $\tau = \epsilon\,t$ and $0 < \epsilon \ll 1$. Then $K = \omega\,J + \epsilon\,(\omega'/\omega)\,J\,\cos\theta\,\sin\theta$, where the prime indicates $\frac{\mathrm{d}}{\mathrm{d}\tau}$. It is then clear that **the energy is no longer constant;** i.e.: $\frac{\mathrm{d}}{\mathrm{d}t}K = \epsilon\,K_\tau = O(\epsilon)$. Further, **neither is J**, as (3.22) shows that $\frac{\mathrm{d}}{\mathrm{d}t}J = -K_\theta = \epsilon\,(\omega'/\omega)\,J\,(\sin^2\theta - \cos^2\theta) = O(\epsilon)$. However,

while the solutions are no longer strictly periodic, they are still "slowly varying" periodic. Hence, **taking the average of J over one period** leads to using the fact that the average over θ of the right hand side in (3.22) vanishes.

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{J} = O(\epsilon^2), \quad (3.24)$$ HAA:eqn:08

An appropriate multiple scales expansion of the solution is:

$$J = J_0 + \epsilon\, J_1(\tau,\, \Psi) + \dots \quad (3.25)$$ HAA:eqn:09

where $\Psi = \frac{1}{\epsilon}\int^{\tau} \omega(s)\,\mathrm{d}s$, J_0 is constant, and

$$\theta = \Psi + \epsilon\, \Theta_1(\tau,\, \Psi) + \dots \quad (3.26)$$ HAA:eqn:10

Then $\bar{J} = J_0 + \epsilon \bar{J}_1 + \dots$ (with the average done over Ψ).

Observation#1. The right hand side in (3.24) does not vanish because the periodicity is not strict.

Observation#2. Because of (3.24), $\bar{J}$ is called an **adiabatic invariant.** ♣

Note 3.7 Generic action variable behavior for slow time dependence: adiabatic invariants. HAA:not:02

Consider a time dependent Hamiltonian. Even if for every frozen time there is a canonical transformation to action-angle variables, when time variation is included the time derivative of the generating function must be added to the transformed Hamiltonian ... which (generically) destroys the action-angle form. (3.27) HAA:eqn:11

Specifically, assume a Hamiltonian system with smooth and slow time dependence, $H(\vec{p},\, \vec{q}, \tau)$, where $\tau = \epsilon\, t$ and $0 < \epsilon \ll 1$. Furthermore, assume that: for each fixed τ-value $H(\vec{p},\, \vec{q}, \tau)$ is **completely integrable**; i.e.: there is a canonical transformation to action-angle variables $(\vec{J},\, \vec{\theta})$, for which $H = H(\vec{J},\, \tau)$. Finally, we assume that all the **solutions are quasi-periodic,** so that the θ_j are true angle variables, and the **dependence on θ_j is $2\,\pi$ periodic.** Note that $\omega_\ell = \omega_\ell(\vec{J},\, \tau) = \partial_{J_\ell} H$ **are the frequencies**, since $\dot{\theta}_j = \omega_j$.

The canonical transformation will, of course, generally depend on τ. Hence, **if the time dependence is not "frozen",** the results in §3.2.1 show that the transformed Hamiltonian will have the form

$$K = H(\vec{J},\, \tau) + \epsilon\, M(\vec{J},\, \vec{\theta},\, \tau), \quad (3.28)$$ HAA:eqn:12

where H is the original Hamiltonian and $\epsilon\, M$ is the term generated by the time derivative of the generating function. Note: M is $2\,\pi$ periodic on each θ_j. Then

13. The energy is no longer constant; $\frac{\mathrm{d}}{\mathrm{d}t}K = \epsilon\, K_\tau = O(\epsilon)$. HAA:itm:01

14. $\frac{\mathrm{d}}{\mathrm{d}t}J_\ell = -\epsilon\, \partial_{\theta_\ell} M$ and $\frac{\mathrm{d}}{\mathrm{d}t}\theta_\ell = \partial_{J_\ell}(H + \epsilon\, M) = \omega_\ell + \epsilon\, \partial_{J_\ell} M$. HAA:itm:02

    Hence the action variables, J_ℓ, are no longer constant, and the angle variables, θ_ℓ, are no longer linear in time. However, *the solutions are still approximately $2\,\pi$-periodic in the θ_ℓ*, hence taking the average over one period in θ_ℓ we obtain That is, $\bar{J}_\ell$ **is an adiabatic invariant.**

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{J}_\ell = O(\epsilon^2). \quad (3.29)$$ HAA:eqn:13

    Note: the average of $\partial_{\theta_\ell} M$ does not vanish because of the lack of perfect periodicity.

This generalizes the results in note 3.6. ♣

3.2.3 Inverted harmonic potential in action-angle variables (example)

Now change $H = \frac{1}{2}(p^2 + \omega^2\, q^2)$ in § 3.2.2 to where $\omega > 0$ is a constant (we will **look at $\omega = \omega(t)$** later).

$$H = \frac{1}{2}\,(p^2 - \omega^2\, q^2), \quad (3.30)$$ IHP:eqn:01

With the Lagrangian $p\,\dot{q} - H$, the equations are $\dot{p} = -H_q = \omega^2\, q$ and $\dot{q} = H_p = p$. Note that **the solutions here are not oscillatory, and the notion of an adiabatic invariant breaks down** (as we will see below). Next we reduce the system to action-angle variables. Let

$$p = \sqrt{2\,\omega\, J}\, \cosh\theta \quad \text{and} \quad q = \sqrt{2\, J/\omega}\, \sinh\theta, \quad (3.31)$$ IHP:eqn:02

with† $J > 0$ and $$K = H = \omega J. \quad (3.32)$$ IHP:eqn:03

Then **(3.31) is canonical, with J the new momentum.**

† Note that (3.31) solves the equations with: $\theta = \omega t + \theta_0$, and J, θ_0 constants.

Proof: $(p\dot{q} - H) - (J\dot{\theta} - K) = \frac{\mathrm{d}}{\mathrm{d}t}(J \cosh\theta \sinh\theta)$.

The **Generating function,** corresponding to (3.12), is $$G = -\frac{1}{2\omega} p^2 \frac{\sinh\theta}{\cosh\theta}. \quad (3.33)$$ IHP:eqn:04

It is straightforward to check that $\mathcal{G} = G + pq = J \cosh\theta \sinh\theta$, $G_p = -q$, and $G_\theta = -J$.

The equations are $$\dot{\theta} = +K_J = \omega, \quad (3.34)$$ IHP:eqn:05

and $$\dot{J} = -K_\theta = 0. \quad (3.35)$$ IHP:eqn:06

Note now that

15. These equations are the same as (3.21–3.22) when $\dot{\omega} = 0$ there. **How can this be?** One system has periodic solutions while the other does not! The **answer is that the canonical transformations are very different:** one is 2π-periodic in θ, while the other is not. IHP:itm:01

16. The (3.31) transformation is **valid in only one of the "quadrants"** produced by the stable/unstable manifolds of the saddle at the origin. **Specifically:** $p > \omega|q|$ — the stable/unstable manifolds are the four level curves $H = 0$, with the origin excluded. Further, unlike the § 3.2.2 case, **the region of validity does not enclose the critical point. Similar transformations apply in the other quadrants.** IHP:itm:02

    It is not possible to have a continuous transformation valid in a region including one of the stable/unstable manifolds, because there is a topology change across them. (3.36) IHP:eqn:07

17. Equation **(3.23) (or similar) no longer applies,** since the orbits are not closed. Hence *a geometrical interpretation of J is not evident* (to me, at least). However, the **interpretation of θ as area along the orbit** (see remark 3.6) **remains valid,** albeit without a way to normalize it‡ — as shown below. IHP:itm:03

    ‡ Note that the normalization of θ is how ω is defined. See item 18.

    Details: Consider the orbit associated with some arbitrary energy $H = E = \omega J > 0$, which is given by $p = \sqrt{2E + \omega^2 q}$ in the quadrant of validity — see item 16. The area "under" the orbit (computed from $q = 0$ to some arbitrary position in the orbit) is then $A(E, q) = \int_0^q \sqrt{2E + \omega^2 x^2}\,\mathrm{d}x$. Hence the area between the orbit and the one corresponding to $E + \mathrm{d}E$ is $$\mathrm{d}A = \left(\int_0^q \frac{1}{\sqrt{2E + \omega^2 x^2}}\,\mathrm{d}x\right)\mathrm{d}E = \frac{1}{\omega}\theta\,\mathrm{d}E, \quad (3.37)$$ IHP:eqn:08 where the substitution $x = \frac{\sqrt{2E}}{\omega}\sinh\phi$ has been used. This is, basically, the same as the formula for A_ϕ obtained in the proof of remark 3.6; but now a "comparison" area (i.e.: $A_{\mathcal{R}}$) is not available.

18. There is an **important difference with the meaning of ω for (3.17) as compared with (3.30).** When a periodic solutions exist, there is a time scale that is selected by the solutions, the period, which then determines ω. On the other hand, when there is no specific time scale associated with the solutions, one can select the time scale arbitrarily (which then makes ω arbitrary as well). For example (3.16) with $\alpha = \beta = 1/\omega$ and $\gamma = 1/\omega^2$ transforms H in (3.30) into $K = \frac{1}{2}(P^2 - Q^2)$. IHP:itm:04

19. A final **important difference occurs when a slowly varying $\omega = \omega(\tau)$ is introduced** ($\tau = \epsilon t$ and $0 < \epsilon \ll 1$). **Then J is not an adiabatic invariant:** the argument in note 3.6, leading to (3.24), fails IHP:itm:05

because there is no period over which to average. $\boldsymbol{J}$ changes at the same rate $\boldsymbol{O(\epsilon)}$ as the energy or the frequency, not one order higher!

3.2.4 Action-angle variables for generic 2D Hamiltonian

Consider a Hamiltonian system in 2D, $\dot{\boldsymbol{q}} = \boldsymbol{H_p}$ and $\dot{\boldsymbol{p}} = -\boldsymbol{H_q}$, in a neighborhood $\boldsymbol{\mathcal{D}}$ of a **(strict, local) minimum (or maximum) of $\boldsymbol{H}$.** In $\boldsymbol{\mathcal{D}}$ the solutions are periodic, with orbits given by the level curves of $\boldsymbol{H}$. There we introduce **local coordinates $\boldsymbol{J}$ and $\boldsymbol{\theta}$ (action-angle variables),** as explained in figure 3.1. Next we show that **the transformation $\boldsymbol{(q,\, p) \to (\theta,\, J)}$ is canonical, with** $\boldsymbol{\dot\theta = H_J,\ \ \dot J = 0,}$ (3.39) AAG:eqn:02 and $\boldsymbol{H = H(q,\, p) = H(J)}$. (3.38) AAG:eqn:01

20. From the definitions in figure 3.1, it should be clear that the energy $\boldsymbol{E}$, the enclosed area $\boldsymbol{A}$, and the action $\boldsymbol{J}$ are all equivalent variables parameterizing the orbits. Furthermore; $\boldsymbol{A > 0}$ always, while: AAG:itm:01
    For a minimum: $\boldsymbol{E > H_{\min}}$ and $\boldsymbol{J, \omega < 0}$. **For a maximum: $\boldsymbol{E < H_{\max}}$** and $\boldsymbol{J, \omega > 0}$.
    The **exception** to this is the maximum/minimum itself, a critical point of the Hamiltonian where $\boldsymbol{J = 0}$ and $\boldsymbol{\theta}$ is not defined (coordinate singularity).
21. Let $\boldsymbol{T = 2\,\pi/\omega}$ be the period of the orbit defined by $\boldsymbol{H = E}$. AAG:itm:02
    Then $$\boldsymbol{\frac{dJ}{dE} = \frac{1}{\omega}.}\quad (3.40)$$ AAG:eqn:03
    Proof. $\boldsymbol{T = \int_0^T dt = \oint (1/H_p)\, dq}$. Thus $\boldsymbol{\frac{dJ}{dE} = \frac{1}{2\,\pi}\oint p_E\, dq = \frac{T}{2\,\pi} = \frac{1}{\omega}}$, where we use that $\boldsymbol{H = E}$ defines $\boldsymbol{p = p(q,\, E)}$ (multiple valued), with $\boldsymbol{p_E = 1/H_p}$.
    Note that **(3.40) is the same as $\boldsymbol{\frac{dH}{dJ} = \omega}$, which is the first equation in (3.39).**
    The second equation is trivial, since (by construction) $\boldsymbol{J}$ is a constant for each orbit.
22. At This point it should be clear that we can AAG:itm:03
    write $\boldsymbol{q = q(\theta, J)}$ and $\boldsymbol{p = p(\theta, J)}$, where $$\boldsymbol{\omega\, q_\theta = H_p \quad \text{and} \quad \omega\, p_\theta = -H_q.}\quad (3.41)$$ AAG:eqn:04
    Then, **to show that the transformation is canonical,**
    it is enough to show that $\boldsymbol{[p\, dq - H] - [J\, d\theta - H]}$ is a perfect differential. That is: there is a function $\boldsymbol{\Psi}$ such that $\boldsymbol{d\Psi = p\, dq - J\, d\theta = p\, q_J\, dJ + (p\, q_\theta - J)\, d\theta}$. Assuming that $\boldsymbol{\mathcal{D}}$ is **simply connected**, this amounts to checking that $\boldsymbol{(p\, q_J)_\theta = (p\, q_\theta - J)_J}$. This follows from (3.40) and (3.41), since: $\boldsymbol{(p\, q_J)_\theta - (p\, q_\theta - J)_J = p_\theta\, q_J - p_J\, q_\theta + 1 = -\frac{1}{\omega}(H_q\, q_J + H_p\, p_J) + 1 = -\frac{1}{\omega}\frac{dH}{dJ} + 1 = 0}$. ♣

Remark 3.7 Explanation for figure 3.1 **(geometrical meaning of the action and the angle).** Consider the situation near a minimum (say, located at the origin) of the Hamiltonian $\boldsymbol{H = H(p,\, q)}$. Then: AAG:rem:01

(1) The level curves, $\boldsymbol{H = E > H_{\min}}$, are orbits, tracked clockwise — left panel, where $\boldsymbol{\nabla H = (H_q,\, H_p)}$, $\boldsymbol{\vec F = (H_p,\, -H_q)}$ is the flow vector, and $\boldsymbol{E}$ is a constant.

(2) If $\boldsymbol{A}$ = area enclosed by a level curve (left panel), then define $$\boldsymbol{J = \frac{1}{2\,\pi}\oint p\, dq = -\frac{1}{2\,\pi}A,}$$ **where $\boldsymbol{J}$ is the action.**

(3) Let $\boldsymbol{A_a}$ be the area of the annular region between two infinitesimally close level curves (right panel). Let $\boldsymbol{A_\theta}$ be the fraction of $\boldsymbol{A_a}$, measured counterclockwise along the orbit, starting from some fixed/arbitrary "radial" curve (in the picture, the positive q-axis). Then **the angle variable is defined by** $\boldsymbol{\theta = 2\,\pi\, A_\theta/A_a.}$

(4) Because Hamiltonians preserve area in phase space, **along orbits $\boldsymbol{\theta}$ is linear in time:** $\boldsymbol{\dot\theta = \omega,}$ where $\boldsymbol{\omega = \omega(J)}$ **is the angular frequency** (negative for a minimum).

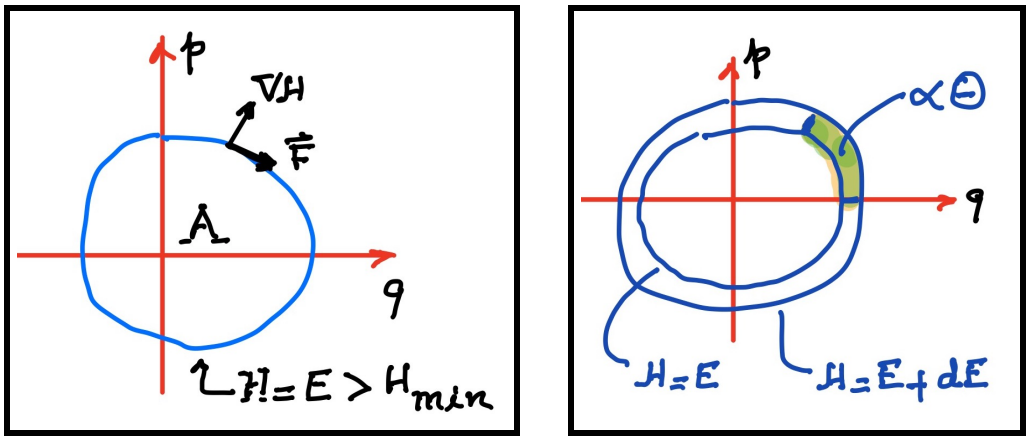

Figure 3.1: **Geometrical meaning of the action and the angle.** See remark 3.7.

Note#1. An orbit is uniquely identified by either $\boldsymbol{E}$ or $\boldsymbol{J}$.

Note#2. The **maximum** case is similar, with **signs reversed:** $\boldsymbol{E < H_{\max}}$, $\boldsymbol{J = \frac{1}{2\pi} A}$, and $\boldsymbol{\omega > 0}$. ♣

Remark 3.8 The transformation is generally not global. The transformation in (3.38) is defined in a neighborhood $\mathcal{D}$ of the critical point. How large can $\mathcal{D}$ be? Well, if the Hamiltonian has any saddles, then the transformation cannot be valid any further than the related saddle connections — see (3.36) in item 16. For that matter, even if there are no saddles, but there is more than one center (i.e.: maximum or minimum of $\boldsymbol{H}$), then there will be orbits separating the centers across which the transformation will be singular.† Hence, generally **global only if there is a single center, no saddles.** AAG:rem:02

† Example: $\boldsymbol{H = \frac{q}{(1+q^2)(1+p2)}}$ has a maximum at $(q, p) = (1, 0)$ and a minimum at $(q, p) = (1, 0)$, nothing else. Then the transformations above yield:

(a) $\boldsymbol{J > 0}$ for $\boldsymbol{q > 0}$, with $\boldsymbol{J \to +\infty}$ as the limit orbit $\boldsymbol{q \equiv 0}$ is approached.

(b) $\boldsymbol{J < 0}$ for $\boldsymbol{q < 0}$, with $\boldsymbol{J \to -\infty}$ as the limit orbit $\boldsymbol{q \equiv 0}$ is approached. ♣

Remark 3.9 Generic reduction to action-angle variables. It is interesting that AAG:rem:03

(proof at the remark's end) **the transformation $(q, p, H) \longleftrightarrow (\theta, J, H)$ is canonical,** (3.42) AAG:eqn:05

where (a) $\boldsymbol{J = J(E)}$ is an arbitrary function of the energy $\boldsymbol{E = H}$, with $\boldsymbol{J' = dJ/dE \neq 0}$;

(b) $\boldsymbol{\theta = \tilde{\omega}\, t}$, where $\boldsymbol{\tilde{\omega} = 1/J'}$ and the time dependence is given by the solutions. A curve in phase space, cutting the orbits once, must be selected. There $\boldsymbol{\theta = 0}$.

The **transformations in § 3.2.2 and § 3.2.3 are exactly (3.42), with $J = E/\omega$.** This *choice is special for* § 3.2.2 because it is the one that makes $\boldsymbol{J}$ an adiabatic invariant, and selects $\tilde{\boldsymbol{\omega}}$ as the true angular frequency. On the other hand, as explained there, there does not seem to be any special reason for $\boldsymbol{J = E/\omega}$ in § 3.2.3. Further, **(3.38) is also the special case of (3.42), where $\tilde{\omega}$ is the true angular frequency.** Finally, **(3.42) also yields reductions to action-angle variables near saddles** (generalizing § 3.2.3). However, the *same issues pointed out in items 17–19 arise.* An **exception to this occurs when $\boldsymbol{H}$** is periodic in $\boldsymbol{q}$ (e.g.: pendulum, see example 3.1-f), and all the orbits (except for saddle connections) are periodic. Then an angular frequency can be properly defined, and formulas like $\boldsymbol{J = \frac{1}{2\pi} \oint p\, dq}$ make sense.

Proof. By construction 3.40–3.41) are valid. Thus the proof in item 22 applies. ♣

Remark 3.10 Even though (3.42) appears to be global, there is **no contradiction with remark 3.8,** because the orbits themselves are singular at critical points, saddle connections, etc. As an example, it is enough to look at (3.31), which yields $p = \omega\,|q|$ only by taking the limit $|\theta| \to \infty$, with $J = e^{-2\,|\theta|} \to 0$. ♣ AAG:rem:04

Example 3.1 **Particle in a potential:** $H = \frac{1}{2}p^2 + V(q)$, e.g.: (a) harmonic oscillator: $V = \frac{1}{2}\,q^2$; (b) pendulum: $V = (1 - \cos q)$; etc. Then, near a local minimum† of V, $V_m = V(q_m)$, AAG:exa:01 AAG:eqn:06

$$\frac{1}{\omega} = \frac{1}{\pi}\int_{q_l}^{q_r} \frac{1}{p}\,\mathrm{d}q, \quad J = \frac{1}{\pi}\int_{q_l}^{q_r} p\,\mathrm{d}q, \quad \text{and} \quad \theta = \omega \int^{q} \frac{1}{p(x,\,E)}\mathrm{d}x, \tag{3.43}$$

where $p = p(q,\,E) = \sqrt{2\,(E - V(q))}$, $V_m < E < V_M$ is the energy, $V_M = V(q_M)$ is nearest local maximum, q_l is the nearest root of $V(q) = E$ to the left of q_m, and q_r is the nearest root to the right.

† For simplicity, we assume that V is smooth and $V'' \neq 0$ at points where $V' = 0$.

Note that:

(a) We replaced $\oint$ by one sided integrals, hence the $\frac{1}{2\,\pi}$ factors have become $\frac{1}{\pi}$.

(b) The equation for θ is valid for only half a period, as q traverses from q_m to q_M.

(c) The transformation becomes singular as E approaches V_M, since $\frac{\mathrm{d}J}{\mathrm{d}E} = \frac{1}{\omega}$, and $\omega \to 0$ (period goes to infinity) in this limit.

(d) For $E = V_M$ the orbit becomes a saddle connection — maximums of V yield saddles of H.

(e) Note that the **formulas remain valid beyond V_M if the potential still confines the solutions to be periodic.** Specifically, they will hold in any range $E_m < E < E_M$ where $V(q) = E$ has only two solutions, $q_l < q_r$, and $E > V(q)$ for $q_l < q < q_r$. Example: consider $V = (1 - \cos q) + \frac{1}{4}\,q^2$.

(f) Assume now that V is periodic in q, of period T_q. Then for $E > \max(V)$ we can use AAG:eqn:07

$$\frac{1}{\omega} = \frac{1}{2\,\pi}\int_0^{T_q} \frac{1}{p}\,\mathrm{d}q, \quad J = \frac{1}{2\,\pi}\int_0^{T_q} p\,\mathrm{d}q, \quad \text{and} \quad \theta = \omega \int_0^{q} \frac{1}{p(x,\,E)}\mathrm{d}x. \tag{3.44}$$

In this case, except for saddle connections, all orbits are included within the range of one of the possible canonical transformations to action-angle variables, and the associated J's become adiabatic invariants when the potential is slowly varying. ♣

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4 Quadratic Hamiltonians (linear equations)

The Hamiltonian for a linear homogeneous system has the form

$$H = \frac{1}{2}\,\vec{z}^{\,T} A \vec{z}, \tag{4.1}$$ QuHa:eqn:01

where $\vec{z} = (\vec{q}\,;\ \vec{p})$, and A is a $2d \times 2d$ symmetric matrix. **The associated Lagrangian is**

$$L = \tfrac{1}{2}\,\vec{z}^{\,T} \mathcal{J}^T \tfrac{\mathrm{d}}{\mathrm{d}t}\vec{z} - H, \tag{4.2}$$ QuHa:eqn:02

where $\mathcal{J}$ is defined below.
Important: **unless otherwise stated, in this subsection all matrices are real.**

Definition 4.1 The $2d \times 2d$ matrix $\mathcal{J}$ is given by
$\mathcal{J} = [0,\, I_d;\, -I_d,\, 0]$, where $I_d = d \times d$ identity. Note $\mathcal{J}^T = -\mathcal{J}$ and $\mathcal{J}^2 = -1$. (4.3) QuHa:eqn:03

Notation. (1) We use a semicolon to stack vertically and a comma to align horizontally.
(2) By default vectors are column.
(3) For any square matrices M_1 and M_2,
diag$(M_1,\, M_2) = [M_1,\, 0;\, 0,\, M_2]$ and odig$(M_1,\, M_2) = [0,\, M_1;\, M_2,\, 0]$.
Thus $\vec{z}$ above has size $2d \times 1$, $\vec{q}$ and $\vec{p}$ are $d \times 1$, and $\mathcal{J} =$ odig$(I_d,\, -I_d)$. ♣

The system equations are $\dfrac{\mathrm{d}}{\mathrm{d}t}\vec{z} = \mathcal{J}\, A\, \vec{z}$. (4.4) QuHa:eqn:04

Remark 4.1 Restrictions are needed on A. CL:rem:01
As pointed out in remark 3.4, Hamiltonian systems are too broad a class. In particular, an arbitrary linear system $\frac{\mathrm{d}}{\mathrm{d}t}\vec{q} = B\,\vec{q}$, B a $d \times d$ matrix, can be embedded into the Hamiltonian with $H = \vec{p}^T B\,\vec{q}$, which has the equations $\frac{\mathrm{d}}{\mathrm{d}t}\vec{q} = B\,\vec{q}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\vec{p} = -B^T\,\vec{p}$, and correspond, in (4.1), to $A = [0,\, B^T;\, B,\, 0]$.
Thus **we make the following assumptions about A:**

r1 **Either A is positive definite, $A > 0$.**

r2 **Or $A = [B,\, 0;\, 0,\, K]$, B and $K > 0$ symmetric, nonsingular, $d \times d$ matrices.**

When r1 applies, we show below that the system is equivalent to d uncoupled harmonic oscillators. That is: the **system is completely integrable, oscillatory.** **Case r2 corresponds** to $H = \frac{1}{2}\,(\vec{p}^T K\,\vec{p} + \vec{q}^T B\,\vec{q})$; i.e.: the linearization (near an equilibrium state) of a "standard" classical mechanics Hamiltonian ($H =$ kinetic energy + potential energy), with a generic kinetic energy $\frac{1}{2}\vec{p}^T K\,\vec{p}$. This system is **also completely integrable,** but it may include inverted harmonic potential components — see § 3.2.3.
In **both cases A is nonsingular,** to eliminate degenerate subspaces of critical points ($\frac{\mathrm{d}}{\mathrm{d}t}\vec{z} = 0$ when $A\,\vec{z} = 0$). **Switching to $A < 0$ in r1** works too, but gives nothing new (it amounts to $t \to -t$). **However there are other interesting subclasses not listed here, e.g.: r3** — where "interesting" means: reducible to the behaviors of $2\,d$ Hamiltonians.

r3 **A commutes with $\mathcal{J}$. That is: $A = [C,\, B;\, -B,\, C]$, with $B^T = -B$ and $C^T = C$.**

Then $\mathcal{J}\,A$ is anti-symmetric, and the system has fundamentally oscillatory behavior, same as what happens when item **r1** applies (analyzing this case is left to the reader). ♣

4.1 Canonical transformations for linear systems

A linear transformation with $U(t) = 2d \times 2d$ matrix, **is canonical iff U is symplectic** (see § 4.1.2). Then where the updated matrix $\tilde{A}$ **is symmetric**

$$\vec{z} \to \vec{y} = U\,\vec{z}, \quad H \to K = H - \tfrac{1}{2}\,\vec{y}^T \mathcal{J}\,\dot{U}\,U^{-1}\vec{y}, \tag{4.5}$$ CL:eqn:01

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\vec{y} &= \mathcal{J}\,\tilde{A}\,\vec{y}, & \text{(a)} \\ \tilde{A} &= (U^{-1})^T A U^{-1} - \mathcal{J}\,\dot{U}\,U^{-1}, & \text{(b)} \end{cases} \tag{4.6}$$ CL:eqn:02

— as shown in (4.21). To show this substitute
$\vec{y} = \boldsymbol{U}\,\vec{z}$ into (4.4), and use (4.13). That (4.5) is canonical iff $\boldsymbol{U}$ is symplectic is proved in item 24.

Further details, and proofs:

23. **Let $\dot{U} = 0$.** Then: **$\tilde{A}$ is nonsingular iff A is nonsingular,** and **$\tilde{A} > 0$ iff $A > 0$.** CL:itm:01
This is trivial. **However, this cannot be guaranteed if $\dot{U} \neq 0$.** Example: let $\vec{u}$ be a solution of (4.4), and select the time dependence of $\boldsymbol{U}$ such that $\vec{v} = \boldsymbol{U}\,\vec{u} =$ nonzero constant. Then (4.6-a) yields $\tilde{\boldsymbol{A}}\,\vec{v} = \boldsymbol{0}$. This requires **only** the ability to specify the target of $\boldsymbol{U}$ at one vector for every time. In 2d this is trivial, for then symplectic reduces to $\det(\boldsymbol{U}) = 1$. Thus take the harmonic oscillator where the solutions trace circles at constant angular speed, and select $\boldsymbol{U}$ as a rotation that cancels this motion.

24. **Now we prove that the transformation (4.5) is canonical iff U is symplectic.** CL:itm:02
(a) If $\boldsymbol{U}$ is symplectic, the difference of the Lagrangians is a time derivative (hence the transformation is canonical). Using (4.6-b), (4.13), and (4.18): $\boldsymbol{L_1 - L_2} = \frac{1}{2}\,\vec{z}^{\,T}\boldsymbol{\mathcal{J}}^{\,T}\frac{\mathrm{d}}{\mathrm{d}t}\vec{z} - \frac{1}{2}\,\vec{y}^{\,T}\boldsymbol{\mathcal{J}}^{\,T}\frac{\mathrm{d}}{\mathrm{d}t}\vec{y} - \frac{1}{2}\,\vec{y}^{\,T}\boldsymbol{\mathcal{J}}\,\dot{\boldsymbol{U}}\,\boldsymbol{U}^{-1}\vec{y} = \boldsymbol{0}$.
(b) To prove the reverse, assume $\dot{\boldsymbol{U}} = \boldsymbol{0}$, since the time dependence is absorbed by the change in the Hamiltonian. Then note that $\boldsymbol{L_1 - L_2} = \frac{1}{2}\,\vec{z}^{\,T}\boldsymbol{\mathcal{J}}^{\,T}\frac{\mathrm{d}}{\mathrm{d}t}\vec{z} - \frac{1}{2}\,\vec{y}^{\,T}\boldsymbol{\mathcal{J}}^{\,T}\frac{\mathrm{d}}{\mathrm{d}t}\vec{y} = \vec{z}^{\,T}\boldsymbol{\mathcal{S}}\frac{\mathrm{d}}{\mathrm{d}t}\vec{z}$, where $\boldsymbol{2\,\mathcal{S}} = \boldsymbol{\mathcal{J}}^{\,T} - \boldsymbol{U}^{T}\boldsymbol{\mathcal{J}}^{\,T}\boldsymbol{U}$ is skew symmetric. Hence $\boldsymbol{L_1 - L_2}$ cannot be a time derivative unless $\boldsymbol{\mathcal{S}} = \boldsymbol{0}$, which is equivalent to (4.18). Thus $\boldsymbol{U}^{\boldsymbol{T}}$, hence $\boldsymbol{U}$, is symplectic.

4.1.1 Properties of the matrix $\mathcal{J}A$, and reduction to normal form

Here $\boldsymbol{\mathcal{J}} = [\boldsymbol{0},\ \boldsymbol{I_d};\ \boldsymbol{-I_d},\ \boldsymbol{0}]$ and $\boldsymbol{I_d} = d \times d$ identity matrix (actually, any $\boldsymbol{\mathcal{J}}$ satisfying (4.3) can be used, but we will stick to this form). We assume **$\boldsymbol{A}$ is as in remark 4.1** — i.e.: it is symmetric, and satisfies either **r1** or **r2**. We also assume **$\boldsymbol{A}$ = constant** — no point in having $\boldsymbol{A} = \boldsymbol{A}(t)$ here. We will show that: **When r1 applies, there is a canonical transformation reducing (4.4) to a set of decoupled harmonic oscillators** — this is Williamson's theorem [3, 1]. **When r2 applies, there is a canonical transformation reducing (4.4) to a set of decoupled Hamiltonians of the form $H = \frac{1}{2}(p_j^2 + \gamma_j\,q_j^2)$, with $\gamma_j \neq 0$.**

Theorem 4.1 **Case: item r1 in remark 4.1 applies;** i.e.: $\boldsymbol{A}$ is symmetric and positive definite. Then PJA:thm:01
there is a canonical transformation $\vec{y} = \boldsymbol{U}\,\vec{z}$ that reduces the Hamiltonian (4.1) to the normal form

$$\boldsymbol{H = \tfrac{1}{2}\,\textstyle\sum_{j=1}^{d}\omega_j\,(P_j^2 + Q_j^2),} \qquad (4.7)$$ PJA:eqn:01

for some frequencies $\boldsymbol{\omega_j > 0}$, where $\vec{\boldsymbol{y}} = (\vec{\boldsymbol{Q}};\ \vec{\boldsymbol{P}})$.
This is equivalent to (see (4.6))

$$\tilde{\boldsymbol{A}} = (\boldsymbol{U}^{-1})^{\boldsymbol{T}}\boldsymbol{A}\boldsymbol{U}^{-1} = \mathrm{diag}(\boldsymbol{D},\ \boldsymbol{D}), \qquad (4.8)$$ PJA:eqn:02

where $\boldsymbol{D} = \mathrm{diag}(\boldsymbol{\omega_1},\ \ldots,\ \boldsymbol{\omega_d})$.
Note that **(4.7) is equivalent to**

$$\boldsymbol{H = \tfrac{1}{2}\,\textstyle\sum_{j=1}^{d}(\tilde{p}_j^2 + \omega_j^2\,\tilde{q}_j^2),} \qquad (4.9)$$ PJA:eqn:03

via the canonical transformation $\tilde{\boldsymbol{p}}_j = \sqrt{\omega_j}\,\boldsymbol{P}_j$ and
$\boldsymbol{Q}_j = \sqrt{\omega_j}\,\tilde{\boldsymbol{q}}_j$ (that this transformation is canonical follows from $\tilde{\boldsymbol{p}}_j\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\boldsymbol{q}}_j = \boldsymbol{P}_j\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{Q}_j$; see § 3.2.1). ♣

Proof of theorem 4.1. Let $\boldsymbol{M = \mathcal{J}\,A}$, and define the **scalar product** $\langle\,\vec{\boldsymbol{x}}_1,\ \vec{\boldsymbol{x}}_2\,\rangle_{\boldsymbol{A}} = \vec{\boldsymbol{x}}_1^{\boldsymbol{T}}\boldsymbol{A}\,(\vec{\boldsymbol{x}}_2)^*$ — here * denotes the **complex conjugate,** and the $\vec{\boldsymbol{x}}_n$ are **$\boldsymbol{2\,d}$ vectors, not necessarily real.** It is easy to see that **$\boldsymbol{M}$ is skew-adjoint with respect to this scalar product** (it is also non-singular). Hence

p1 There is an eigenvector base $\{\vec{\boldsymbol{v}}_j\}_1^{2\,d}$ such that [1] $\boldsymbol{M\vec{v}_j = i\omega_j\vec{v}_j}$ and [2] $\langle\,\vec{\boldsymbol{v}}_n,\ \vec{\boldsymbol{v}}_m\,\rangle_{\boldsymbol{A}} = \boldsymbol{\delta_{nm}}$, where the **$\boldsymbol{\omega_j \neq 0}$ are all real.**

p2 M real $\Rightarrow \{-\omega_j\}_1^{2\,d} = \{\omega_j\}_1^{2\,d}$. Hence can arrange things so that, for $\mathbf{1} \le n \le d$, [3] $\omega_n = -\omega_{n+d} > 0$ and [4] $\vec{v}_{n+d} = (\vec{v}_n)^*$.
Detail: [2] follows by selecting an orthonormal basis for each eigenspace [eigenspaces belonging to different eigenvalues are orthogonal]. Hence once a base for some $\lambda = i\,\omega$ is selected, use the conjugate base for $\overline{\lambda} = -i\,\omega$ to obtain [4].

p3 For $\mathbf{1} \le n \le d$, let $\vec{a}_n = \mathrm{Re}(\vec{v}_n)$ and $\vec{b}_n = \mathrm{Im}(\vec{v}_n)$. Then [5] $M\,\vec{a}_n = -\omega_n\,\vec{b}_n$ and $M\,\vec{b}_n = \omega_n\,\vec{a}_n$, which is equivalent to [1].
Further, [2] is equivalent to [6] $\langle \vec{a}_n,\, \vec{a}_m \rangle_A = \langle \vec{b}_n,\, \vec{b}_m \rangle_A = \frac{1}{2}\,\delta_{nm}$ and $\langle \vec{a}_n,\, \vec{b}_m \rangle_A = 0$.
Note that this implies that $\{\vec{a}_n,\, \vec{b}_n\}$ is an orthogonal base.
Detail: Define $\{\vec{a}_n,\, \vec{b}_n\}$ beyond $n = d$. Then: [7] $\vec{a}_{n+d} = \vec{a}_n$, and $\vec{b}_{n+d} = -\vec{b}_n$,
while the real and imaginary parts of [2] are: [8a] $\delta_{nm} = \langle \vec{a}_n,\, \vec{a}_m \rangle_A + \langle \vec{b}_n,\, \vec{b}_m \rangle_A$
and [8b] $\langle \vec{a}_n,\, \vec{b}_m \rangle_A = \langle \vec{a}_m,\, \vec{b}_n \rangle_A$.
Hence, using [7], [6] follows (inspect [8] for $(n,\, m)$ and $(n,\, m+d)$, $\mathbf{1} \le n,\, m \le d$).

p4 Multiply $\vec{a}_n$ and $\vec{b}_n$ by $\sqrt{2\,\omega_n}$, so [6] becomes [9] $\langle \vec{a}_n,\, \vec{a}_m \rangle_A = \langle \vec{b}_n,\, \vec{b}_m \rangle_A = \omega_n\,\delta_{nm}$ and $\langle \vec{a}_n,\, \vec{b}_m \rangle_A = 0$.

p5 Define the $2\,d \times 2\,d$ matrix V by [10] $V = [\vec{a}_1,\, \ldots,\, \vec{a}_d,\, \vec{b}_1,\, \ldots,\, \vec{b}_d]$.
Then [5] becomes [11] $M\,V = V\,\mathcal{J}\,\mathrm{diag}(D,\, D)$, with $D = \mathrm{diag}(\omega_1,\, \ldots,\, \omega_d)$,
while [9] is the same as [12] $V^T A\,V = \mathcal{D}$, where $\mathcal{D} = \mathrm{diag}(D,\, D)$.

p6 Now: $(V\,\mathcal{J})^{-1}\mathcal{J}\,A\,V = (V\,\mathcal{J})^{-1} M\,V = (V\,\mathcal{J})^{-1} V\,\mathcal{J}\,\mathcal{D} = \mathcal{D} = V^T A\,V$, using [11–12]. But $A\,V$ is nonsingular; hence $V^T = (V\,\mathcal{J})^{-1}\mathcal{J} = -\mathcal{J} V^{-1}\mathcal{J}$, or $V^T \mathcal{J}\,V = \mathcal{J}$. That is: V **is symplectic.** Take now $U = V^{-1}$, then [12] is (4.8). **QED**

Theorem 4.2 **If item r2 in remark 4.1 applies,** $A = [B,\, 0;\, 0,\, K]$ ($K > 0$, B, symmetric, non-singular, $d \times d$ matrices), **there is a canonical transformation** $\vec{y} = U\,\vec{z}$ **that reduces the (4.1) to the normal form** PJA:thm:02

$$H = \tfrac{1}{2} \textstyle\sum_{j=1}^{d} (P_j^2 + \mu_j\, Q_j^2), \quad (4.10)$$ PJA:eqn:04

for some constants $\mu_j \neq 0$, where $\vec{y} = (\vec{Q};\, \vec{P})$.
This is equivalent to (see (4.6)) PJA:eqn:05

$$\tilde{A} = (U^{-1})^T A U^{-1} = \mathrm{diag}(D,\, I_d), \quad (4.11)$$

where $D = \mathrm{diag}(\mu_1,\, \ldots,\, \mu_d)$ — $\mu_j \neq 0$ real.
Note that **(4.7) is equivalent to** PJA:eqn:06

$$H = \tfrac{1}{2} \textstyle\sum_{j=1}^{d} \sqrt{|\mu_j|}\,(\tilde{p}_j^2 + \sigma_j\,\tilde{q}_j^2), \quad (4.12)$$

where $\sigma_j = \mathrm{sign}(\mu_j)$. This via the canonical transformation $P_j = \sqrt[4]{|\mu_j|}\,\tilde{p}_j$ and $\tilde{q}_j = \sqrt[4]{|\mu_j|}\,Q_j$ (that this transformation is canonical follows from $\tilde{p}_j \frac{\mathrm{d}}{\mathrm{d}t}\tilde{q}_j = P_j \frac{\mathrm{d}}{\mathrm{d}t} Q_j$; see § 3.2.1).
Note: Strictly speaking, B non-singular is not needed to obtain (4.10). In this case some of the μ_j may vanish (corresponding to "free" particles). The transformation to (4.12) then fails. ♣

Proof of theorem 4.2. A (4.24) canonical transformation preserves the form of A, with $B \to B_* = C^T B\,C$ and $K \to K_* = C^{-1} K\,(C^{-1})^T$. Below we reduce the system matrix to the form in (4.11) by successive transformations of this type:

T1 Select $C_1 =$ orthogonal matrix such that $K_1 = \text{diag}(k_1, \ldots, k_d)$, where $k_j > 0$.

T2 Select $C_2 = \text{diag}(\sqrt{k_1}, \ldots, \sqrt{k_d})$. This yields $K_2 = I_d$.

T3 Select $C_3 =$ orthogonal matrix such that $B_3 = \text{diag}(\mu_1, \ldots, \mu_d)$, where the μ_j are the real, nonzero, eigenvalues of B_2. Then $K_3 = K_2 = I_d$. **QED**

Note 4.1 **Eigenvalues of $\mathcal{J}A$ in the absence of restrictions beyond $A = A^T$ nonsingular. If λ is an eigenvalue of $\mathcal{J}A$ (thus $\lambda \neq 0$), then so are $-\lambda$ and $\pm\overline{\lambda}$ — all with equal eigenspace structure.** This is **"the best" we can say,** as examples like the one in remark 4.1 show, with $\mathcal{J}A = \text{diag}(B, -B^T)$ and B an arbitrary matrix. **Recall:** $\text{diag}(M_1, M_2) = [M_1, 0; 0, M_2]$ and $\text{odig}(M_1, M_2) = [0, M_1; M_2, 0]$. PJA:not:01

Proof. Since $(\mathcal{J}A)^T = -A\mathcal{J}$, $\mathcal{J}A$ and $A\mathcal{J}$ have the same Jordan normal form, with the **eigenvalue signs reversed.** Further, $\mathcal{J}$ maps $A\mathcal{J}$ Jordan boxes to $\mathcal{J}A$ Jordan boxes: $(A\mathcal{J})\vec{v}_n = \lambda\vec{v}_n + \vec{v}_{n-1} \;\to\; (\mathcal{J}A)\vec{u}_n = \lambda\vec{u}_n + \vec{u}_{n-1}$, with $\vec{u}_n = \mathcal{J}\vec{v}_n$. It follows that if λ is an eigenvalue of $\mathcal{J}A$, then so is $-\lambda$ (with the same Jordan box structure). Furthermore: since $\mathcal{J}$ and A are nonsingular, $\lambda \neq 0$. The result for $\pm\overline{\lambda}$ now follows because $\mathcal{J}A$ is real. ♣

4.1.2 Properties of symplectic matrices

Definition: a $2d \times 2d$ **matrix U is symplectic iff** $\mathcal{J} = U\,\mathcal{J}\,U^T$, (4.13) PSM:eqn:01

where $\mathcal{J}$ is defined† in (4.3). Let the **1-st (resp. 2-nd) d rows of U be $\vec{r}_n^T$ (resp. $\vec{s}_n^T$).** Then

$$(4.13) \iff \beta(\vec{r}_n, \vec{r}_m) = \beta(\vec{s}_n, \vec{s}_m) = 0 \;\text{ and }\; \beta(\vec{r}_n, \vec{s}_m) = \delta_{nm}, \quad (4.14)$$ PSM:eqn:02

where **β is the skew-symmetric non-degenerate bilinear form** $\beta(\vec{x}, \vec{y}) = \vec{x}^T\mathcal{J}\vec{y}$. (4.15) PSM:eqn:03

Because of (4.18), **the columns of U also satisfy (4.14).**

Equation (4.18) also shows that **U is symplectic iff $\beta(U\vec{x}, U\vec{y}) = \beta(\vec{x}, \vec{y})$ for all $\vec{x}$ and $\vec{y}$.** (4.16) PSM:eqn:04

† Actually, any $\mathcal{J}$ satisfying (4.3) can be used, but we will stick to this form. Except for results such as (4.14), everything else applies to the generic case. In fact, it all depends on (4.16); any skew-symmetric non-degenerate bilinear form corresponds to a $\mathcal{J}$, and viceversa.

Symplectic matrices have the following properties:

- **A symplectic matrix is invertible, with:** $U^{-1} = -\mathcal{J}U^T\mathcal{J}$. (4.17) PSM:eqn:05

Proof. Left multiply the right hand side by U. Then use (4.3) and (4.13).

- **The transpose of a symplectic matrix is symplectic:** $\mathcal{J} = U^T\mathcal{J}U$. (4.18) PSM:eqn:06

Proof. Substitute (4.17) into $\mathcal{J} = \mathcal{J}U^{-1}U$.

- **Both the inverse, and the negative, of a symplectic matrix are symplectic.** (4.19) PSM:eqn:07

Proof. For the inverse, invert (4.18) and use $\mathcal{J}^{-1} = -\mathcal{J}$. That $-U$ is symplectic is trivial.

- **The product of symplectic matrices is symplectic.** (obvious). (4.20) PSM:eqn:08

- Let $U = U(t)$ be a C^1 family of symplectic matrices.

Then: **$\dot{U}\mathcal{J}U^T$ and $\mathcal{J}\dot{U}U^{-1}$ are symmetric.** (4.21) PSM:eqn:09

Proof. From (4.13), $0 = \dot{U}\mathcal{J}U^T + U\mathcal{J}\dot{U}^T \;\Rightarrow\; \dot{U}\mathcal{J}U^T$ is symmetric $\Rightarrow \mathcal{J}(\dot{U}\mathcal{J}U^T)\mathcal{J}^T = \mathcal{J}\dot{U}U^{-1}(U\mathcal{J}U^T)\mathcal{J}^T = \mathcal{J}\dot{U}U^{-1}$ is symmetric (use (4.13) for the last equality).

- **$\mathcal{J}$ and the identity are symplectic** (obvious). (4.22) PSM:eqn:10

● **A 2×2 matrix U is symplectic iff $\det U = 1$.**† Four cases are possible: (4.23) PSM:eqn:11

c1 The eigenvalues of U are non-real, complex conjugates, on the unit circle.

c2 The eigenvalues of U are real, with $\lambda_1 = \lambda_2^{-1} \neq 1$.

c3 $U = \pm$ identity.

c4 U has a (single) repeated eigenvalue $\lambda = \pm 1$, with geometric multiplicity 1.
This shows that **a symplectic matrix need not be diagonalizable.**

† That (4.13) is equivalent to $\mathbf{det}(\boldsymbol{U}) = \mathbf{1}$ is a straightforward calculation.

● **Examples of symplectic matrices produced via quadratic generating functions** are displayed below, where $\boldsymbol{C}$ is a $\boldsymbol{d} \times \boldsymbol{d}$ nonsingular matrix. It is easy to check that **all these cases satisfy (4.13).**

Recall that: $\mathrm{diag}(\boldsymbol{M_1}, \boldsymbol{M_2}) = [\boldsymbol{M_1}, \mathbf{0}; \mathbf{0}, \boldsymbol{M_2}]$ and $\mathrm{odig}(\boldsymbol{M_1}, \boldsymbol{M_2}) = [\mathbf{0}, \boldsymbol{M_1}; \boldsymbol{M_2}, \mathbf{0}]$.

g1 Case (3.12), $\boldsymbol{G = -\vec{p}^{\,T} C\, \vec{Q}}$. Then:
$\vec{q} = -G_{\vec{p}} = +C\,\vec{Q}$, and $\vec{P} = -G_{\vec{Q}} = +C^T \vec{p}$, so that $\boldsymbol{U = \mathrm{diag}(C^{-1}, C^T)}$. (4.24) PSM:eqn:12

g2 Case (3.13), $\boldsymbol{G = +\vec{p}^{\,T} C\, \vec{P}}$. Then:
$\vec{q} = -G_{\vec{p}} = -C\,\vec{P}$, and $\vec{Q} = +G_{\vec{P}} = +C^T \vec{p}$, so that $\boldsymbol{U = \mathrm{odig}(C^T, -C^{-1})}$. (4.25) PSM:eqn:12

g3 Case (3.14), $\boldsymbol{G = -\vec{q}^{\,T} C\, \vec{Q}}$. Then:
$\vec{p} = +G_{\vec{q}} = -C\,\vec{Q}$, and $\vec{P} = -G_{\vec{Q}} = +C^T \vec{q}$, so that $\boldsymbol{U = \mathrm{odig}((C^{-1})^T, -C)}$. (4.26) PSM:eqn:14
This is the same as (4.25), with $C \to C^{-1}$.

g4 Case (3.15), $\boldsymbol{G = +\vec{q}^{\,T} C\, \vec{P}}$. Then:
$\vec{p} = +G_{\vec{q}} = +C\,\vec{P}$, and $\vec{Q} = +G_{\vec{P}} = +C^T \vec{q}$, so that $\boldsymbol{U = \mathrm{diag}(C^T, C^{-1})}$. (4.27) PSM:eqn:15
This is the same as (4.24), with $C \to (C^{-1})^T$.

g5 The **examples** above are **not the most general (quadratic) generating functions,** by far. Following note 3.3, the general case involves a quadratic function of $\vec{s}$ and $\vec{S}$, not just the special cases above. Furthermore, even for the special cases shown, the selected $\boldsymbol{G}$ is not the most general possible. For example, in item **g1**, the most general choice is $\boldsymbol{G = -\vec{p}^{\,T} C\, \vec{Q} + \vec{p}^{\,T} C_p\, \vec{p} + \vec{Q}^{\,T} C_Q\, \vec{Q}}$, for some matrices $\boldsymbol{C_p}$ and $\boldsymbol{C_Q}$.

g6 To continue with item **g5**: **Any symplectic matrix can be produced via a generating function.**
This is, of course, the linear version of note 3.2.
Proof. Let $\boldsymbol{U}$ be a symplectic matrix, associated with the canonical transformation $\vec{Z} = \boldsymbol{U}\,\vec{z}$. Pick $\boldsymbol{d}$ columns of $\boldsymbol{U}$, w.l.o.g. the first $\boldsymbol{d}$. These columns must be linearly independent, since $\boldsymbol{U}$ is invertible. This means that $\boldsymbol{d}$ rows of these columns must be linearly independent; w.l.o.g. assume that these are the first $\boldsymbol{d}$ rows. But this implies that we can write $\vec{q}$ in terms of $\vec{Q}$ and $\vec{p}$. Hence $\vec{p}$ and $\vec{Q}$ are a system of coordinates in phase space, and the argument in note 3.2 applies.

● **Properties of the spectrum of a symplectic matrix U.** If $\boldsymbol{\lambda}$ is an eigenvalue of $\boldsymbol{U}$, then so are $\overline{\boldsymbol{\lambda}}$, $\mathbf{1}/\boldsymbol{\lambda}$, and $\mathbf{1}/\overline{\boldsymbol{\lambda}}$. Furthermore, they all have the exact same eigenspace structure; i.e.: Jordan boxes. In particular $\mathbf{det}(\boldsymbol{U}) = \mathbf{1}$. Note that there are no restrictions beyond this, as examples like (4.24) illustrate.
Proof. This is obvious for $\overline{\boldsymbol{\lambda}}$ (and $\mathbf{1}/\overline{\boldsymbol{\lambda}}$ once proved for $\mathbf{1}/\boldsymbol{\lambda}$). Now, $\mathbf{1}/\boldsymbol{\lambda}$ is an eigenvalue of $(\boldsymbol{U^{-1}})^{\boldsymbol{T}}$, with the same eigenspace structure as $\boldsymbol{\lambda}$. However, from (4.17), it follows that $(\boldsymbol{U^{-1}})^{\boldsymbol{T}} = \boldsymbol{J^{-1} U J}$. Hence $\mathbf{1}/\boldsymbol{\lambda}$ is an eigenvalue of $\boldsymbol{U}$,

with the same eigenspace structure as $\boldsymbol{\lambda}$.

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5 Examples

5.1 Korteweg-de Vries (KdV) equation

Here we consider the a-dimensional (linear) KdV equation

$$\boldsymbol{u_t + c\,u_x - \gamma\,u_{xxx} = 0}, \qquad (5.1)$$ 01:eqn:01

where $\boldsymbol{c}$ and $\boldsymbol{\gamma}$ are constants. The sinusoidal solutions to this equation have the form

$$\boldsymbol{u = a\,\cos(k\,x - \omega\,t + \theta_0)}, \qquad (5.2)$$ 01:eqn:02

with $\boldsymbol{\omega = c\,k + \gamma\,k^3}$. Thus it is a dispersive equation. It "must" have an associated Lagrangian. To find it, let $\boldsymbol{u = \phi_x}$; then

$$\boldsymbol{\phi_{xt} + c\,\phi_{xx} - \gamma\,\phi_{xxxx} = 0}. \qquad (5.3)$$ 01:eqn:03

Why this? The reason is that the Lagrangian would have to be a quadratic function of the solution and its derivatives. Then the corresponding Euler-Lagrange equations would only involve terms with an even number of derivatives — e.g.: see (2.14). Hence the switch from (5.1) to (5.3). ♣

Then

$$\boldsymbol{L = -\tfrac{1}{2}\,\phi_x\,\phi_t - \tfrac{1}{2}\,c\,\phi_x^2 - \tfrac{1}{2}\,\gamma\,\phi_{xx}^2}, \qquad (5.4)$$ 01:eqn:04

with Euler-Lagrange equations

$$\boldsymbol{\partial_t L_{\phi_t} + \partial_x L_{\phi_x} - \partial_x^2 L_{\phi_{xx}} = 0}, \qquad (5.5)$$ 01:eqn:05

yields (5.1). Substituting $\boldsymbol{\phi = a\,\cos\theta}$ – see (1.1) – and averaging over $\boldsymbol{\theta}$, yields the **average Lagrangian**

$$\boldsymbol{\mathcal{L} = \tfrac{1}{4}\,a^2\,\left(\omega\,k - c\,k^2 - \gamma\,k^4\right)}. \qquad (5.6)$$ 01:eqn:06

From this the following equations follow:

Dispersion relation; $\boldsymbol{\mathcal{L}_a = 0}$.

$$\boldsymbol{(\omega - c\,k - \gamma\,k^3)\,k = 0}. \qquad (5.7)$$ 01:eqn:07

Note 5.1 Equation (5.7) has the spurious solution $\boldsymbol{k = 0}$, 01:not:01
originating from the fact that ϕ is a potential. On the other hand, (5.1) *admits* $\boldsymbol{u =}$ *constant as a solution*, which has a "slowly varying" version $\boldsymbol{u \sim \overline{u}(X,T)}$, with $\boldsymbol{X = \epsilon\,x}$, $\boldsymbol{T = \epsilon\,t}$, and $\boldsymbol{\overline{u}_T + c\,\overline{u}_X = 0}$. In the context of (5.3) this corresponds to a "pseudo-phase" $\boldsymbol{\beta = \frac{1}{\epsilon}\,\Psi(X,T)}$, with $\boldsymbol{\overline{u} = \Psi_X}$. For the nonlinear KdV the dynamics of the mean value $\overline{u}$ becomes important because it couples with the modulated wave [2]. *The phenomena of a wave coupling with a mean is fairly common in systems where a "mean" exists. The first observation of this is the "Stokes drift", which is triggered by the nonlinear corrections in water waves (and was first computed by Stokes).* ♣

Conservation of wave action,
$\boldsymbol{\partial_T \mathcal{L}_\omega - \partial_X \mathcal{L}_k = 0}$. i.e.:

$$\boldsymbol{\partial_T\left(\tfrac{1}{4}\,k\,a^2\right) + \partial_X\left(\tfrac{1}{4}(-\omega + 2\,c\,k + 4\,\gamma\,k^3)\,a^2\right) = 0}. \qquad (5.8)$$ 01:eqn:08

However, since $\boldsymbol{\omega = c\,k + \gamma\,k^3}$, and $\boldsymbol{c_g = c + 3\,\gamma\,k^2}$, this is the same as

$$\boldsymbol{\partial_T\left(\tfrac{1}{4}\,k\,a^2\right) + \partial_X\left(c_g\,\tfrac{1}{4}\,k\,a^2\right) = 0}. \qquad (5.9)$$ 01:eqn:09

Conservation of energy, with $\boldsymbol{\mathcal{E} = \omega\,\mathcal{L}_\omega = \frac{1}{4}\,\omega\,k\,a^2}$

$$\boldsymbol{\partial_T\,\mathcal{E} + \partial_X(c_g\,\mathcal{E}) = 0}. \qquad (5.10)$$ 01:eqn:10

Note 5.2 Energy is defined in terms of the conservation law associated with the fact that $\boldsymbol{L}$ is time independent, §2.1.1. Thus the energy density is $\boldsymbol{E = \phi_t\,L_{\phi_t} - L = \frac{1}{2}\left(c\,\phi_x^2 + \gamma\,\phi_{xx}^2\right) = \frac{1}{2}\left(c\,u^2 + \gamma\,u_x^2\right)}$, whose average is $\boldsymbol{\mathcal{E}}$. There are other conservation laws; for example $\boldsymbol{u^2}$ is conserved. Infinitely more can be generated by observing that any derivative of $\boldsymbol{u}$ satisfies the KdV equation; hence $\boldsymbol{u_x^2}$, $\boldsymbol{u_t^2}$, $\boldsymbol{u_{xx}^2}$, ... are conserved. However, only one is associated with the time symmetry, and has the physical meaning energy. ♣ 01:not:02

5.2 Schrödinger equation, and classical limit of QM (YET TO BE DONE).

5.3 Wave equation and the Eikonal equation

Here we consider the a-dimensional wave equation
$$u_{tt} - \operatorname{div}(c^2\,\nabla u) = 0, \quad (5.11)$$ 03:eqn:01
where $c = c(\vec{x}) = O(1) > 0$. This is not a dispersive wave system, but it is conservative, with Lagrangian
$$L = \tfrac{1}{2}\,u_t^2 - \tfrac{1}{2}\,c^2\,(\nabla u)^2. \quad (5.12)$$ 03:eqn:02
Introduce a high frequency approximation
$$u = a\cos\theta, \quad \theta = \tfrac{1}{\epsilon}\Theta, \quad (5.13)$$ 03:eqn:03
where $a = a(\vec{x},t)$, $\Theta = \Theta(\vec{x},t)$, and define
$$\omega = -\Theta_t, \quad \vec{k} = \nabla\Theta. \quad (5.14)$$ 03:eqn:04
The average Lagrangian is then
$$\mathcal{L} = \tfrac{1}{4}\,a^2\,(\omega^2 - c^2\,k^2). \quad (5.15)$$ 03:eqn:05
This leads to the equations:

Dispersion relation
$$G = \omega^2 - c^2\,k^2 = 0. \quad (5.16)$$ 03:eqn:06
This can also be written in the form
$$\Theta_t^2 = (c\,\nabla\Theta)^2, \quad (5.17)$$ 03:eqn:07
a pde governing the evolution of the phase.

Group speed
$$\vec{c}_g = \tfrac{1}{\omega}c^2\,\vec{k}. \quad (5.18)$$ 03:eqn:08
Since $\|\vec{c}_g\| = c$, $\vec{c}_g$ is a *vector normal to the wave-front of length* c.
Note: This makes no sense for $\vec{k} = 0$. However the equations are derived assuming a-dimensional with $\lambda = O(1)$.

Conservation of wave action
$$(\tfrac{1}{2}\,\omega\,a^2)_t + \operatorname{div}(\vec{c}_g\,\tfrac{1}{2}\,\omega\,a^2) = 0, \quad (5.19)$$ 03:eqn:09
where we have used that $c^2\,\vec{k} = \omega\,\vec{c}_g$.

Conservation of energy
$$(\tfrac{1}{2}\,a^2)_t + \operatorname{div}(\vec{c}_g\,\tfrac{1}{2}\,a^2) = 0, \quad (5.20)$$ 03:eqn:10
which applies because c does not depend on time.
Note: Both/either (5.19–5.20) are referred as the **transport equation.**

5.3.1 Constant frequency and the Eikonal

A situation of interest is that where the waves have a constant frequency:
$$\Theta = \Psi(\vec{x}) - 1. \quad (5.21)$$ 03:eqn:11
Then $\vec{c}_g = c^2\,\vec{k} = c^2\,\nabla\Psi$, and we have the equations:

Equation (5.17) reduces to the **Eikonal equation**
$$(c\,\nabla\Psi)^2 = 1, \quad (5.22)$$ 03:eqn:12
while the **transport equation** becomes
$$(a^2)_t + \operatorname{div}(c^2\,a^2\,\nabla\Psi) = 0. \quad (5.23)$$ 03:eqn:13
These equations can be reduced to a system of ode along bicharacteristics (the light rays in optics) for pseudo-particles (photons, phonons) that carry the wave properties. In terms of the arclength, the equations are 03:eqn:14
$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}s} = c\,\vec{k}, \quad \frac{\mathrm{d}\vec{k}}{\mathrm{d}s} = \nabla\left(\frac{1}{c}\right), \quad \text{and} \quad \frac{\mathrm{d}\Psi}{\mathrm{d}s} = \frac{1}{c}, \quad (5.24)$$
where $\vec{k} = \nabla\,\Psi$ and $\mathrm{d}s = c\,\mathrm{d}t$.
Proof. (1) The first equation is the "definition" of the curves, and the parameter is arclength because the right hand side has unit length as per the Eikonal. (2) For the second: the chain rule gives $\frac{\mathrm{d}k_j}{\mathrm{d}s} = \Psi_{x_j\,x_\ell}\,c\,\Psi_{x_\ell}$,

while taking ∂_{x_j} of the Eikonal yields $0 = c\, c_{x_j}\, \Psi_{x_\ell}^2 + c^2\, \Psi_{x_\ell} \Psi_{x_\ell\, x_j}$. Hence $\frac{\mathrm{d}k_j}{\mathrm{d}s} = -c_{x_j}\, \Psi_{x_\ell}^2 = -c_{x_j}/c^2$.
(3) The third equation follows from the chain rule. ♣
Note that the equations imply $\frac{\mathrm{d}}{\mathrm{d}s} c^2\, k^2 = 2\,(c^2\, k^2 - 1)\,(\vec{k}\cdot\nabla)\, c$. Thus: **if $c^2\, k^2 = 1$ "initially", then it stays this way** — making the equations consistent with the Eikonal.

5.3.2 Fermat's principle

Equation for the rays. Eliminating $\vec{k}$ yields

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{c}\frac{\mathrm{d}\vec{x}}{\mathrm{d}s}\right) = \nabla\left(\frac{1}{c}\right). \quad (5.25)$$ 03:eqn:15

We want to write a variational principle for this equation.
However, using arclength (or time) to parameterize the solutions means that there is a constraint that has to be enforced. It is better to rewrite the equation in terms of some arbitrary parameter τ, so that $\mathrm{d}s = \left|\frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau}\right| \mathrm{d}\tau$. Then

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{c\, v}\frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau}\right) = v\nabla\left(\frac{1}{c}\right). \quad (5.26)$$ 03:eqn:16

where $v = |\mathrm{d}\vec{x}/\mathrm{d}\tau|$. These are the Euler-Lagrange equations for

$$J = \int \frac{v}{c}\,\mathrm{d}\tau. \quad (5.27)$$ 03:eqn:17

Note that $\dfrac{v}{c}\,\mathrm{d}\tau = \dfrac{\mathrm{d}s}{c} = \mathrm{d}t$; hence J **is the travel time for the wave along the ray.** Thus **The rays are stationary paths for the wave travel time.** (5.28) 03:eqn:18
This is **Fermat's principle.**
IMPORTANT. It is **stationary paths, NOT maximums or minimums.** This is a point often left confused in the literature. It is easy to produce examples of rays that either maximize, or minimize, or neither, the wave travel time. There may be an exercise assigned about this.

5.3.3 Caustics, arêtes, singular rays, etc (YET TO BE DONE).

References

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The End.