Lecture 23: Continuous Time Random Walks

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Thus far, we have treated the steps in a random walk as occurring at regular intervals in time, denoted as τ, such that $t = N\tau$. In this lecture, we introduce the concept of a random waiting time, where the variable τ is now sampled from a one-sided probability distribution which we will denote as $\psi(t)$. To address random waiting times in the context of continuous time random walks (CTRWs), we will begin with a brief overview of Laplace transforms and renewal theory, including an example in which the waiting time is given by a Poisson PDF. We will then apply these techniques to a separable CTRW, as first proposed by Montroll and Weiss.

1 Laplace Transforms

Throughout this course, we have used Fourier transforms extensively to describe random walks over infinite spatial domains. For one sided PDFs, such as that describing the waiting time ($\tau > 0$), the Laplace transform is better suited. The forward and inverse Laplace transforms, $\tilde{\psi}(s)$ and $\psi(t)$, respectively, are defined by:

$$\tilde{\psi}(s) = \int_0^\infty e^{-st}\psi(t)\,dt \quad \Leftrightarrow \quad \psi(t) = \int_{c-i\infty}^{c+i\infty} e^{st}\tilde{\psi}(s)\frac{ds}{2\pi i} \tag{1}$$

Here, c denotes a real constant in the half plane of convergence for the transform. The Laplace transform is analogous to the Fourier transform in many ways. Just as the moment generating function, $\hat{p}(k)$, encodes the moments of the PDF $p(x)$, so too does $\tilde{\psi}(s)$ encode the moments of the waiting time (assuming they exist):

$$\langle \tau^n \rangle = \int_0^\infty t^n \psi(t)\,dt = (-1)^n \left.\frac{d^n\tilde{\psi}}{ds^n}\right|_{s=0} \tag{2}$$

A further analogy can be drawn to the Tauberian theorems discussed in Lectures 6 and 7 (2006), where we found that non-analytic terms appeared in the expansion of the CGF at order α, where $p(x) \sim A/t^{1+a}$ $(t \to \infty)$. For the PDFs of the waiting time with this same type of power law tail, we find ($0 < \alpha < 1$):

$$\psi(t) \sim \frac{A}{t^{1+\alpha}} \;(t \to \infty) \quad \Leftrightarrow \quad \tilde{\psi}(s) \sim 1 - Bs^\alpha \;(s \to 0) \tag{3}$$

Furthermore, since the Laplace transform converges for PDFs that diverge as t raised to some positive power, we have the complimentary pair of transforms ($\beta > 0$):

$$\psi(t) \sim \frac{At^{\beta}}{\Gamma(\beta+1)} \ (t \to \infty) \quad \Leftrightarrow \quad \tilde{\psi}(s) \sim \frac{A}{s^{1+\beta}} \ (s \to 0) \tag{4}$$

Finally, we note that Laplace transforms also exhibit the convolution property:

$$[f * g](t) = \int_0^t f(t')g(t-t')\,dt' \quad \Leftrightarrow \quad \left[\widetilde{f * g}\right](s) = \tilde{f}(s)\tilde{g}(s) \tag{5}$$

This is all that we will need to proceed with our discussion of CTRW.

2 Random Waiting Times (Renewal Theory)

Random waiting times have numerous real world applications, including queuing theory. Here, the times at which commands show up at the CPU for execution can be thought of as the events, separated in time by random intervals, τ. Other applications include traffic flow, where τ could represent, for example, the waiting time between taxis.

To begin our discussion of renewal theory, we assume that there are no correlations between steps. We are interested in asking questions about how often events arrive, and thus we define $\mathcal{P}(N,t)$ as the probability of N events occurring in time t, and $\psi(t)$, the PDF of the waiting time. To find $\mathcal{P}(0,t)$, we recognize that this is simply asking the probability that no events have occurred up to time t:

$$\mathcal{P}(0,t) = \int_t^{\infty} \psi(t')\,dt' \equiv \Psi(t) \tag{6}$$

For $\mathcal{P}(1,t)$, we calculate the probability that the first event occurs at time t', and there are no subsequent events. Integrating over all possible values of t' yields:

$$\mathcal{P}(1,t) = \int_0^t \psi(t')\Psi(t-t')\,dt' = \psi * \Psi \tag{7}$$

Generalizing this to N events, we have:

$$\mathcal{P}(N,t) = \psi * \psi * \psi * \ldots * \Psi = \psi^{*N} * \Psi \tag{8}$$

Taking the Laplace transform of this gives us:

$$\tilde{\mathcal{P}}(N,s) = \tilde{\psi}^N \tilde{\Psi} = \frac{\tilde{\psi}(s)^N \left[1 - \tilde{\psi}(s)\right]}{s} \tag{9}$$

We will now use this result to determine the some of the moments of N, the number of steps taken in a particular walk over time t.

2.1 Example: Poisson Process

A Poisson process is defined by the PDF:

$$\psi(t) = \lambda e^{-\lambda t} = \frac{1}{\overline{\tau}} e^{-t/\overline{\tau}} \quad \Leftrightarrow \quad \tilde{\psi}(s) = \frac{\lambda}{\lambda + s} = \frac{1}{1 + \overline{\tau} s} \tag{10}$$

We note from this that $\tilde{\psi}(s) \sim 1 - \overline{\tau} s \ (s \to 0)$, so that $\langle \tau \rangle = \overline{\tau}$. This defines the mean waiting time for the process. From equation (9), we have:

$$\tilde{\mathcal{P}}(N,s) = \left(\frac{\lambda}{\lambda+s}\right)^N \frac{1}{s}\left[1 - \frac{\lambda}{\lambda+s}\right] = \frac{\lambda^N}{(\lambda+s)^{N+1}} \tag{11}$$

$$\mathcal{P}(N,t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[\frac{e^{st}\lambda^N}{(\lambda+s)^{N+1}}\right] ds \tag{12}$$

To evaluate this integral, we deform the contour of integration as shown in figure 1 and apply the residue theorem. Specifically, we use that:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)\,dz}{(z - z_0)^{n+1}} \tag{13}$$

$$\text{Res}(s = -\lambda) = \frac{\lambda^N}{N!} \frac{d^N}{ds^N}\left[e^{st}\right]\Big|_{s=-\lambda} = \frac{(\lambda t)^N e^{-\lambda t}}{N!} \tag{14}$$

From this result, we see that the PDF for *N(t)* is itself a Poisson distribution.

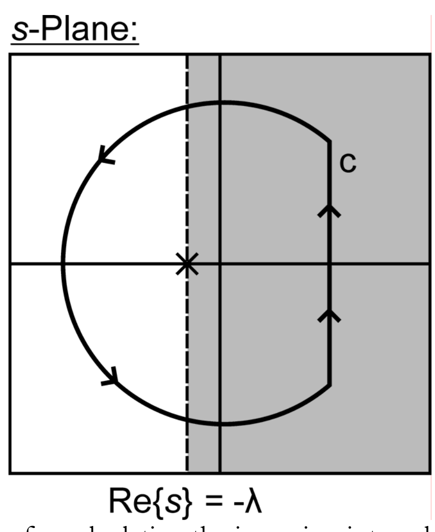

Figure 1: Integration contour, c, for calculating the inversion integral of equation (12).

2.2 Expected Number of Steps

Suppose now that we interpret N as the number of steps taken in a random walk. The expected value of N and its Laplace transform are given by:

$$\langle N\rangle(t)=\sum_{N=0}^{\infty}N\mathcal{P}(N,t) \tag{15}$$

$$\langle\tilde{N}\rangle(s)=\sum_{N=0}^{\infty}N\tilde{\mathcal{P}}(N,s)=\frac{1}{s}\left[1-\tilde{\psi}(s)\right]\sum_{N=0}^{\infty}N\tilde{\psi}(s)^{N} \tag{16}$$

The sum over N can be written as a derivative with respect to $\tilde{\psi}$, giving us:

$$\langle\tilde{N}\rangle(s)=\frac{1}{s}\left[1-\tilde{\psi}(s)\right]\tilde{\psi}(s)\frac{d}{d\tilde{\psi}}\sum_{N=0}^{\infty}\tilde{\psi}(s)^{N} \tag{17}$$

$$\langle\tilde{N}\rangle(s)=\frac{1}{s}\left[1-\tilde{\psi}(s)\right]\tilde{\psi}(s)\frac{d}{d\tilde{\psi}}\left[\frac{1}{1-\tilde{\psi}(s)}\right] \tag{18}$$

$$\langle\tilde{N}\rangle(s)=\frac{\tilde{\psi}(s)}{s\left[1-\tilde{\psi}(s)\right]} \tag{19}$$

Since we are interested in the long time behavior of *N(t)*, we take the inverse transform of equation (???) in the limit as $s\to 0$. There are two cases that we must consider: those in which there is a well defined mean waiting time ($\overline{\tau}=\langle\tau\rangle<\infty$), and those in which the mean waiting time diverges ($\langle\tau\rangle=\infty$).

Case I: $\overline{\tau}=\langle\tau\rangle<\infty$

As long as the mean waiting time is finite, as $s\to 0$, we have $\tilde{\psi}(s)\sim 1-\overline{\tau}s$. Equation (19) then becomes:

$$\langle\tilde{N}\rangle(s)\sim\frac{1-\overline{\tau}s}{s\left[1-1+\overline{\tau}s\right]}\sim\frac{1}{\overline{\tau}s^{2}} \tag{20}$$

The inverse transform of this is $\langle N\rangle(t)=t/\overline{\tau}$. This result justifies our neglect of random waiting times throughout most of this course; provided that a mean waiting time can be defined, the expected number of steps will increment linearly with this characteristic time. An alternate way to see the convergence of a CTRW to a discrete time walk as $t\to\infty$ is to recognize that:

$$\tilde{\mathcal{P}}(N,s)\sim\tau(1-\tau s)^{N}\sim\tau e^{-N\tau s}\quad(N\to\infty,s\to 0) \tag{21}$$

The inverse transform reveals that $\mathcal{P}(N,t)\propto\delta(t-N\tau)$ for $(N\to\infty,s\to 0)$, which is analogous to a DTRW with steps taken at intervals of τ.

Case II: $\langle\tau\rangle=\infty$

The mean waiting time becomes infinite when the waiting time PDF is in the form of equation (3). Equation (19) then becomes:

$$\langle\tilde{N}\rangle(s) \sim \frac{1-Bs^{\alpha}}{s\left[1-1+Bs^{\alpha}\right]} \sim \frac{1}{Bs^{\alpha+1}} \quad (22)$$

which transforms to:

$$\langle N\rangle(t) \sim \frac{t^{\alpha}}{B\Gamma(\alpha+1)} \quad (23)$$

Recalling that $0<\alpha\leq 1$, we see that N scales sublinearly in time. We can extend this further by recalling that $\mathcal{P}(N,t)=\psi^{*N}*\Psi$ and $\tilde{\mathcal{P}}(N,s)=\tilde{\psi}^{N}\tilde{\Psi}$ to see that, as $s\to 0$:

$$\tilde{\mathcal{P}}(N,s) \sim \frac{Bs^{\alpha}}{s}\left(1-Bs^{\alpha}\right)^{N} \underset{(N\to\infty)}{\sim} \frac{Be^{-BNs^{\alpha}}}{s^{1-\alpha}} \quad (24)$$

where we have used that $\left(1+x/N\right)^{N}\sim e^{x}$ as $N\to\infty$. Equation (24) can be expressed in terms of one sided Lévy distributions, the Laplace transforms of which are defined by $\tilde{\ell}_{\alpha,1}(s)=e^{-s^{\alpha}}$. One of the few cases in which the inversion integral can be expressed in terms of elementary functions is when $\alpha=1/2$, for which we find that the Smirnov density emerges once again. The details of this calculation can be found in the notes for lecture 15, 2003.

3 Continuous Time Random Walks (CTRW)

In a continuous time random walk, each step is characterized by a waiting time, τ, and a displacement, Δx. One way to think of this process is a walker that is stationary for a time τ, followed by an instantaneous displacement Δx (a leaper in later terminology). For the present discussion, we will assume that the CTRW is *separable*, in the sense that τ and Δx are independent of each other (in later lectures on leapers and creepers in a more general context, we will relax this constraint). The position of the walker is:

$$X(t)=\sum_{n=0}^{N(t)}\Delta x_{n} \quad (25)$$

We have a choice of two formulations in proceeding: the model of Montroll and Weiss (1965), in which we work primarily with the waiting time PDF, $\psi(t)$, or in terms of $\mathcal{P}(N,t)$. Here, we adopt the former approach and refer the reader to lecture 16, 2003 for a discussion of the latter. The Montroll-Weiss formulation is useful for working with anomalous cases (i.e. diverging mean waiting times), while the formulation in terms of $\mathcal{P}(N,t)$ offers some advantages when the mean waiting time is well defined.

We wish to find the PDF for $X(t)$, $P(x,t)$. This is equal to the summation over all N of the probability that the walker ends up at x in N steps, given that it has taken N steps over time t:

$$P(x,t)=\sum_{N=0}^{\infty}\mathcal{P}(N,t)\cdot P_{N}(x) \quad (26)$$

since the waiting time and step size are independent. Taking the Fourier-Laplace transform gives:

$$
\begin{aligned}
\hat{\tilde{P}}(k,s) &= \sum_{N=0}^{\infty} \tilde{\mathcal{P}}(N,s)\cdot \hat{P}_N(k) \\
&= \frac{1-\tilde{\psi}(s)}{s}\sum_{N=0}^{\infty}\left[\tilde{\psi}(s)\,\hat{p}(k)\right]^N \\
&= \frac{1-\tilde{\psi}(s)}{s}\left[\frac{1}{1-\tilde{\psi}(s)\,\hat{p}(k)}\right]
\end{aligned} \tag{27}
$$

This is the Montroll-Weiss equation. With this result, we can calculate the expected position of the walker, $\langle X\rangle(t)$:

$$
\langle \tilde{X}\rangle(s) = -i\left.\frac{d\hat{\tilde{P}}}{dk}\right|_{\substack{k=0\\ s}} = \frac{\langle\Delta x\rangle\tilde{\psi}(s)}{s\left(1-\tilde{\psi}(s)\right)} \qquad (\text{for } \langle|\Delta x|\rangle < \infty) \tag{28}
$$

Combining this with equation (19), we see that $\langle \tilde{X}\rangle(s) = \langle\Delta x\rangle\langle\tilde{N}\rangle(s)$ and thus $\langle X\rangle(t) = \langle\Delta x\rangle\langle N\rangle(t)$. This result holds even for anomalous cases. We can also determine the second cumulant:

$$
\langle X^2\rangle(t) - \langle X\rangle^2(t) = \sigma_X^2 = \langle N\rangle(t)\sigma_{\Delta x}^2 + \sigma_N^2(t)\langle\Delta x\rangle^2 \tag{29}
$$

where $\sigma_{\Delta x}^2$ denotes the variance in the step size and σ_N^2 denotes the variance in the number of steps taken. Only in the case where there is finite drift ($\langle\Delta x\rangle \neq 0$) does the variability in the number of steps taken contribute to the variance in the walker's position, X. If there is drift, then σ_X^2 can arise even in the case where $\sigma_{\Delta x}^2 = 0$. If a system with this type of behavior is studied from a continuum model, it is impossible to decouple the two contributions by looking at the diffusion coefficient. It is also worth mentioning that the results of (28) and (29) are well known results for random sums of random variables, which closely parallels the processes we are studying here.

3.1 Physical Example

One example of where anomalous CTRWs have been successfully used to model a physical process is DNA gel electrophoresis. This was studied extensively by Weiss in the late 1990's. Very briefly, DNA molecules are forced by an electric field through a porous gel. The rate at which molecules pass through the gel depends on their mobility, a function of size and charge. For relatively small sections of DNA, separation can be achieved in this way. Very large molecules, however, often become trapped in the gel, leading to very long waiting time between displacements. This blurs the bands of different types of DNA molecules significantly, undermining the separation. Interestingly, the work of Weiss suggests that this process is described effectively by a CTRW with finite drift and zero variance in the step size, but a diverging mean waiting time ($\psi(t) \propto t^{-1-\alpha}$ for $(1 < \alpha < 2)$).

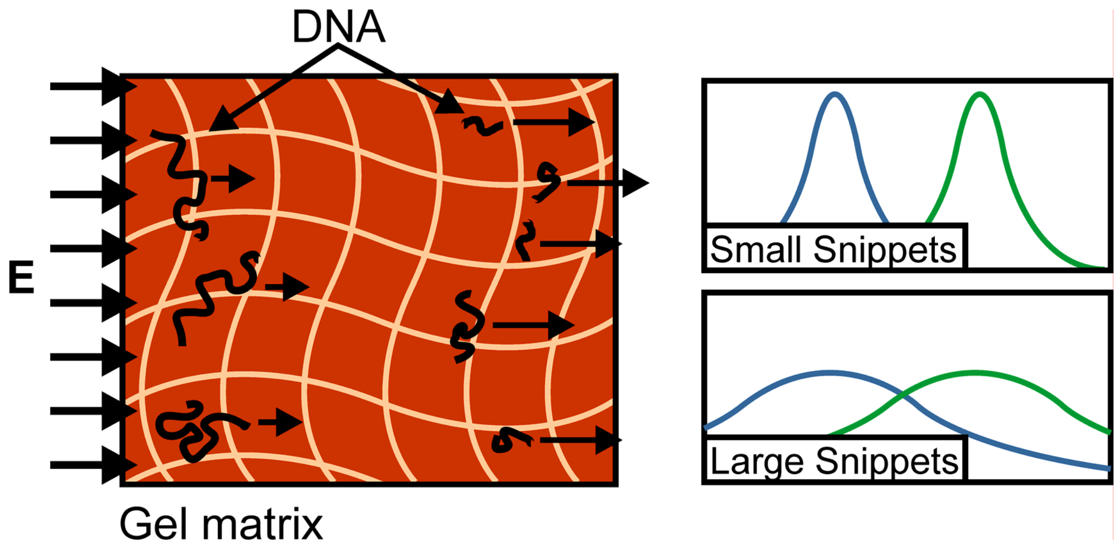

Figure 2: DNA electrophoresis as a physical example where long waiting times emerge. For relatively small snippets of DNA, molecules pass through the gel without becoming trapped for extensive lengths of time. If the molecules become too large, however, they can become entangled at a single location in the gel, leading to anomalous scaling, and poor separation.