Lecture 11: Non-linear Diffusion

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1 Non-linear Drift

In the continuum limit the PDF $\rho(x,t)$ for the position x at time t of a single random walker satisfies the Fokker-Plank equation (FPE), based on a truncation of the Kramers-Moyal expansion

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}\left(D_1(x,t)\rho(x,t)\right) = \frac{\partial^2}{\partial x^2}\left(D_2(x,t)\rho(x,t)\right) \tag{1}$$

In a system of many non-interacting, independent walkers, the FPE will also describe a macroscopic variable, interpreted as the concentration of walkers. If the coefficients D_1 and D_2 depend only on position and time, then we have linear diffusion and consequently a linear PDE, which can then be solved with familiar linear methods.

By contrast, if D_1 and/or D_2 are functions of the probability density (or concentration), then equation (1) is non-linear, and the methods of solution are not as familiar. Three examples of a non-linear FPE are considered in this lecture, illustrating some methods of solution for problems of this type. All three problems are examples where the drift coefficient D_1 is a function of the probability density and the contribution of diffusion is de-emphasized relative to the drift. Examples where the diffusion coefficient D_2 is a function of the probability density will be considered in the next lecture.

1.1 Traffic flow: nonlinear drift but no diffusion

For this example we start with a one-dimensional continuum model, motivated by our experiences in traffic, but without an underlying microscopic model per se. In this case the traffic density along the highway is denoted by $\rho(x,t)$ (*cars/unit-length*). We postulate that a car's velocity u is a function of the traffic density, and so we denote the velocity by $u(x,t) = u(\rho)$ (*unit-length/unit-time*). The flux (equivalently, flow rate) F for the cars is then also a function of the traffic density $F(\rho) \equiv u(\rho)\rho$ (*cars/unit-time*).

The number of cars in a fixed stretch of the highway $x_1 < x < x_2$ is

$$N = \int_{x_1}^{x_2} \rho(x,t)dx$$

and the rate of change of the number of cars in that same stretch of the highway is the difference between the number per unit time that enter at x_1 and the number per unit time that leave at x_2, assuming a "conservation of cars" condition[1]

$$u(x_1,t)\rho(x_1,t) - u(x_2,t)\rho(x_2,t) = -\int_{x_1}^{x_2} \frac{\partial(u(x,t)\rho(x,t))}{\partial x}dx = -\int_{x_1}^{x_2} \frac{\partial F(\rho)}{\partial x}dx$$

[1]Per the title of J.P. Sartre's play No Exit `http://en.wikipedia.org/wiki/No_Exit`. This play is the source of Sartre's famous comment about traffic "l'enfer, c'est les autres."

For any arbitrary stretch of highway (x_1, x_2) the rate of change in the traffic density must balance the divergence of the flux, since we've assumed no source or sink for cars

$$\int_{x_1}^{x_2} \left(\frac{\partial \rho}{\partial t} + \frac{\partial F(\rho)}{\partial x} \right) dx = 0$$

and since the interval (x_1, x_2) is arbitrary, then everywhere

$$\frac{\partial \rho}{\partial t} + \frac{\partial F}{\partial x} = 0 \tag{2}$$

which can also be written by factoring out the derivative of the flux with respect to the traffic density

$$\frac{\partial \rho}{\partial t} + \frac{\partial F}{\partial \rho}\frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0 \tag{3}$$

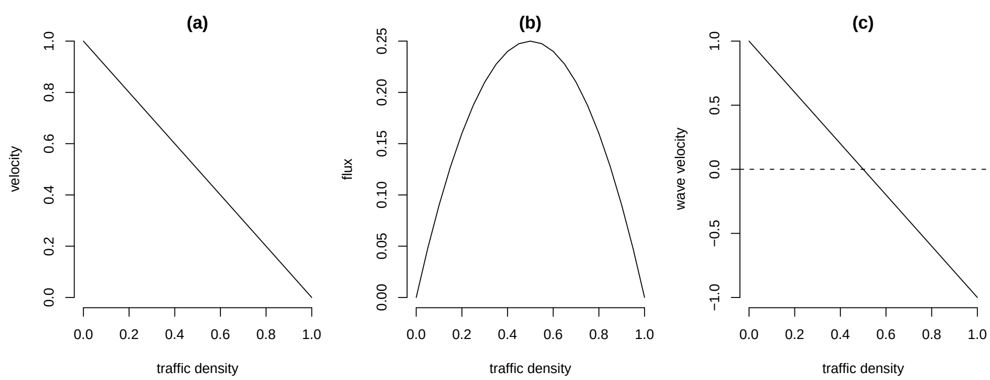
where we identify $c(\rho) = \partial F / \partial \rho$ as a wave velocity.

Equation (3) is an example of the FPE without a diffusion term, and we can find a solution using the method of characteristics (see the Appendix). Two examples are considered in the subsequent sections.

Traffic Flow Example (1) For the following very simple traffic examples, assume that the velocity is linear and decreasing in the traffic density, e.g. of the form $u(\rho) = u_{max}(1 - \rho/\rho_{max})$ (Figure 1(a)). The flux is therefore $F(\rho) = \rho u(\rho)$ (Figure 1(b)), and the wave velocity is $c(\rho) = dF(\rho)/d\rho = u_{max}(1 - 2\rho/\rho_{max})$ (Figure 1(c)).

Figure 1: Traffic

The charts below show the assumptions for the velocity, flux and wave velocity as a function of traffic density for the subsequent traffic problem examples. The parameters u_{max} and ρ_{max} are both assumed to be 1.



Along the characteristic lines for equation (3) we have

$$\frac{dt}{dr} = 1, \quad \frac{dx}{dr} = c(\rho), \quad \frac{d\rho}{dr} = 0$$

and assuming the following initial traffic density

$$\rho(x, 0) = \frac{1}{4}\left(1 + \frac{3e^{-\alpha x}}{1 + e^{-\alpha x}}\right)$$

we can compute the characteristic lines

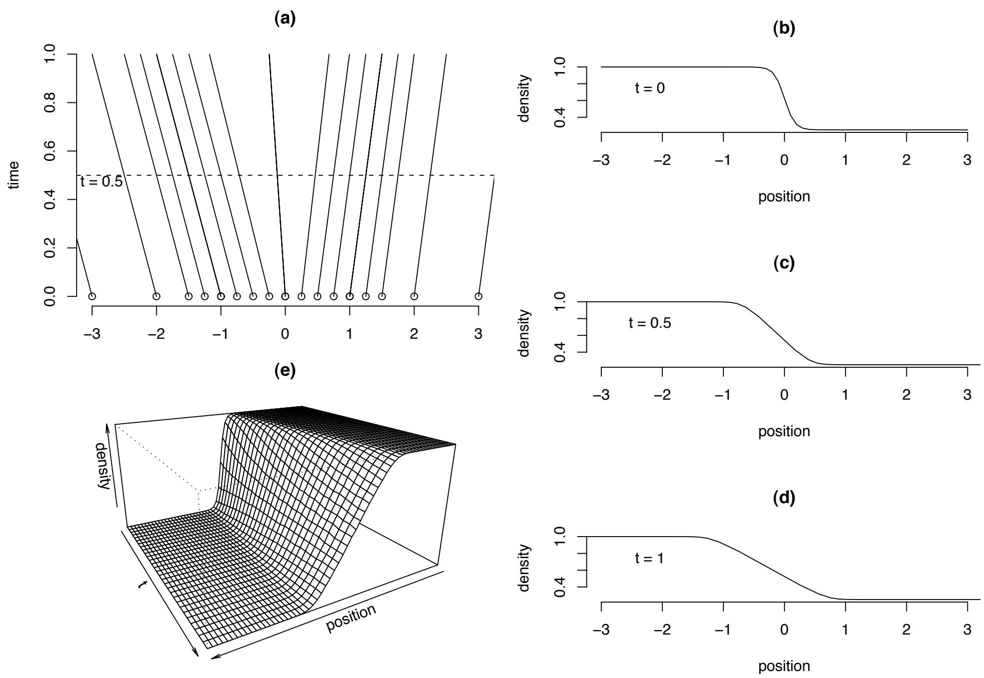
$$x(t) = u_{max}\left(1 - \frac{1}{2\rho_{max}}\left(1 + \frac{3e^{-\alpha x}}{1 + e^{-\alpha x}}\right)\right)t + x \tag{4}$$

which are shown in Figure 2(a).

The initial density can be projected out along the characteristic lines to see how the density evolves over time and the result is shown in Figure 2. Since the traffic velocity decreases with its density, an initial density profile that decreases from left to right means that the cars in front travel faster, so the initial distribution will spread out.

Figure 2: Evolution of Traffic Density

Figure **(a)** shows the characteristic lines as a function of initial position, assuming initial traffic density $\rho(x,0) = \frac{1}{4}\left(1+\frac{3e^{-\alpha x}}{1+e^{-\alpha x}}\right)$ where $\alpha = 12$. The characteristic lines are described by equation (4), and for the given initial density they do not intersect. Figures **(b)**, **(c)**, and **(d)** shows the traffic density over time, at intervals of 0.5 time units. Since the velocity decreases with the traffic density, the cars in the back move more slowly than the cars in the front, and the initial density will spread out over time. Figure **(e)** shows the evolution of the traffic density for $t \in [0,1]$ and $x \in [-3,-3]$.



Traffic Flow Example (2) A quite different behaviour is shown in the next example. Assuming the following initial traffic density

$$\rho(x,0) = \begin{cases} 0.25, & |x| > 1.5 \\ 1 - |x|/2, & |x| \leq 1.5 \end{cases} \tag{5}$$

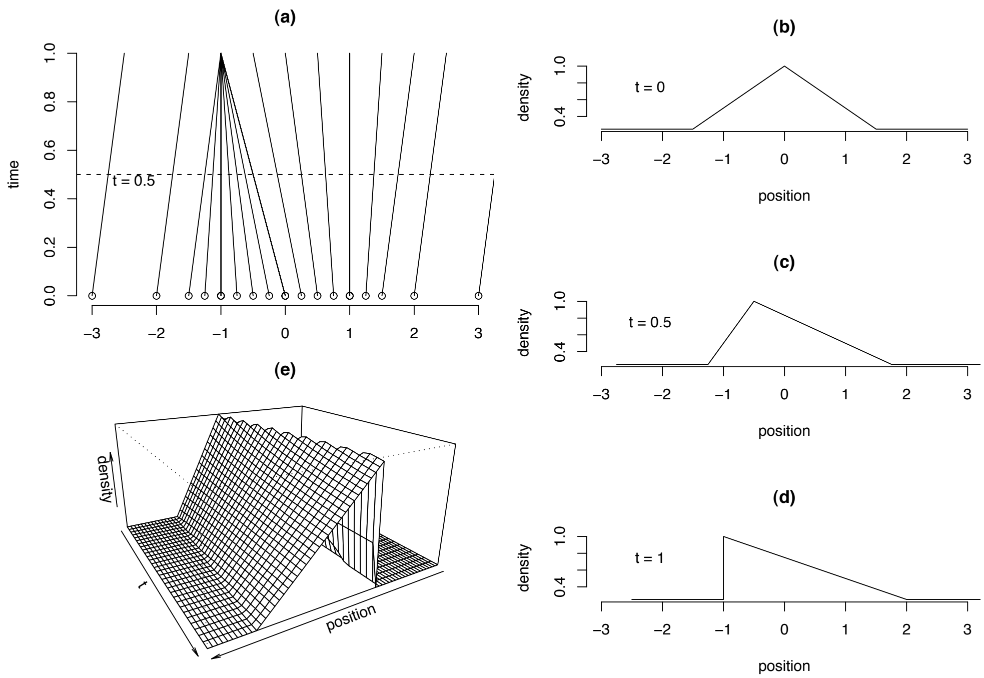
we can again compute the characteristic lines

$$x(t) = \begin{cases} u_{max}\frac{2\rho_{max}-1}{2\rho_{max}}t + x, & |x| > 1.5 \\ u_{max}\left(1 - \frac{2-|x|}{\rho_{max}}\right)t + x, & |x| \leq 1.5 \end{cases}$$

but this case the characteristic lines that originate in $-1.5 \leq x \leq 0$ will intersect at $t^* = 1$, as illustrated in Figure 3(a).

Figure 3: Developing Shocks in the Traffic Problem - Initial Triangular Density

Figure **(a)** shows the characteristic lines as a function of position for an initial triangular traffic density, as given by equation (5). Figures **(b)**, **(c)**, and **(d)** show the developing shock in the traffic flow, at intervals of 0.5 time units. Since the cars in the middle of the initial density travel the slowest, the lower density leading edge pulls away, while the lower density trailing edge catches up, eventually forming a shock. The traffic density is shown at three different times. Figure **(e)** also shows the developing shock for $t \in [0, 1]$ and $x \in [-3, -3]$.



Traffic flow: nonlinear drift with diffusion

A more sophisticated version of the traffic model might consider drivers to have a look-ahead capability, sensing and responding to the gradient of the traffic density ahead of them. Presumably the drivers would slow down if the density was increasing, and otherwise they would speed up. In this version of the model the traffic flux becomes

$$F(\rho) = u(\rho)\rho - D\frac{\partial \rho}{\partial x}$$

and the conservation law of equation (2) is then, for constant diffusivity D

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial F}{\partial x} &= 0 \\ \frac{\partial \rho}{\partial t} + \frac{\partial u(\rho)\rho}{\partial x} &= D\frac{\partial^2 \rho}{\partial x^2} \end{aligned}$$

in this case we may have a “diffusive regularization” of any shock, where the diffusive flux component counteracts the advection flux.

1.2 Fluid Dynamics: non-linear drift

Another example of macroscopic behaviour based on discrete physical models is a simple model for gas dynamics, described by Burgers' Equation [4]

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial \rho}{\partial x} = D \frac{\partial^2 \rho}{\partial x^2} \tag{6}$$

Burgers originally proposed equation (6) as a model for turbulence [2, 3] and it has found application in gas dynamics [5, 7] and acoustics [9], among other areas.

One of the earliest known solutions to Burgers' equation is the Fourier series solution attributed to Fay ([6])

$$\rho(x,t) = -2D \sum_{n=1}^{\infty} \frac{\sin(nx)}{\sinh(nDt)}$$

and for $Dt \ll 1$ the solution has the characteristic sawtooth profile given by the Fourier series

$$x - \pi = -2 \sum_{n=1}^{\infty} \frac{\sin nx}{n}, \quad 0 < x < 2\pi$$

The solitary wave solution (for constants p and α)

$$\rho(x,t) = -Dp \tanh \frac{p}{2}(x + \alpha)$$

can be verified directly.

The Cole-Hopf transformation [5, 7] can be used to solve Burgers' equation using solutions to the linear diffusion equation, as follows. Let $\rho = -\partial h/\partial x$ for some $h(x,t)$ and then let $\psi(x,t) \equiv e^{h(x,t)/2D}$ so that $h(x,t) = 2D \ln \psi(x,t)$ and

$$\frac{\partial h}{\partial t} = \frac{2D}{\psi}\frac{\partial \psi}{\partial t}, \qquad \frac{\partial h}{\partial x} = \frac{2D}{\psi}\frac{\partial \psi}{\partial x}, \qquad \frac{\partial^2 h}{\partial x^2} = \frac{2D}{\psi^2}\left(\psi \frac{\partial^2 \psi}{\partial x^2} - \left(\frac{\partial \psi}{\partial x}\right)^2\right)$$

Then ψ satisfies

$$\frac{\partial \psi}{\partial t} = D \frac{\partial^2 \psi}{\partial x^2} + \frac{F}{2D}\psi$$

(i.e. $h_t = D h_{xx} + \frac{1}{2}(h_x)^2 + F$) for some constant F. If we now define $\psi \equiv e^{Ft/2D}\phi$, then

$$\frac{\partial \phi}{\partial t} = D \frac{\partial^2 \phi}{\partial x^2} \tag{7}$$

whose solutions are familiar. Substituting these solutions back through, we have

$$\rho = -\frac{2D}{\phi}\frac{\partial \phi}{\partial x} \tag{8}$$

For example, the solution to equation (7) for a point source at the origin

$$\phi(x,t) = \sqrt{\frac{\pi}{Dt}} e^{-x^2/4Dt}$$

yields $\rho(x,t) = x/t$. Similarly, the solution to equation (7) for the sum of two point sources at $\pm\pi$ is

$$\phi(x,t) = \sqrt{\frac{\pi}{Dt}} \left(e^{-(x-\pi)^2/4Dt} + e^{-(x+\pi)^2/4Dt}\right)$$

yields the solution attributed to Khokhlov ([10])

$$\rho(x,t) = \frac{x}{t} - \frac{\pi}{t}\tanh \frac{\pi x}{2Dt}$$

1.3 Surface Growth

A model for the evolution of a growing surface is described by the Kardar-Parisi-Zhang equation [8]. In the one-dimensional from of this equation, the height of the surface above a horizontal base is denoted by $h(x,t)$. The model includes both diffusive motion along the surface, in addition to surface growth in a direction normal to the surface.

In the context of this surface growth problem, we identify the density/concentration term from the earlier case with the variation in surface height with space, i.e. let $\rho = -\partial h/\partial x$ and then starting with Burgers' equation

$$\frac{\partial \rho}{\partial t} + \lambda \rho \frac{\partial \rho}{\partial x} = D \frac{\partial^2 \rho}{\partial x^2} \tag{9}$$

we substitute for ρ

$$\begin{aligned} -\frac{\partial^2 h}{\partial x \partial t} + \lambda \frac{\partial h}{\partial x}\left(\frac{\partial^2 h}{\partial x^2}\right) &= -D\frac{\partial^3 h}{\partial x^3} \\ -\frac{\partial^2 h}{\partial x \partial t} + \frac{\lambda}{2}\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right)^2 &= -D\frac{\partial^3 h}{\partial x^3} \end{aligned}$$

Integrating back over x gives, to within a constant of integration, an equation for the surface height

$$\frac{\partial h}{\partial t} = D\frac{\partial^2 h}{\partial x^2} + \frac{\lambda}{2}\left(\frac{\partial h}{\partial x}\right)^2 + F \tag{10}$$

where we identify F with the deposition flux, D with the surface relaxation coefficient, and $h(x,t)$ with the height of the surface. For the choice $F = \lambda + v$, corresponding to a constant velocity of the surface, we have

$$\frac{\partial h}{\partial t} = D\frac{\partial^2 h}{\partial x^2} + \lambda\left(1 + \frac{1}{2}\left(\frac{\partial h}{\partial x}\right)^2\right) + v \tag{11}$$

Other algebraic solutions of equation (10) can be found by first solving for $\rho(x,t)$, i.e. solving Burgers' equation.

KPZ Equation: nonlinear drift but no diffusion

As in the case of traffic flow, we can first look at solutions to the KPZ equation for cases where there is no diffusion, and we consider two cases for the initial condition. In the first case, the initial surface height is convex.

Convex surface, no diffusion If the initial height is convex and consists of two linear segments, e.g.

$$h(x,0) = \begin{cases} h_0 + \alpha x, & x \le 0 \\ h_0 + \beta x, & x \ge 0 \end{cases}$$

then $\rho(x,t) = -\partial h/\partial x$ has initial condition

$$\rho(x,0) = \begin{cases} -\alpha, & x < 0 \\ -\beta, & x > 0 \end{cases}$$

and the characteristic lines are

$$x(t) = \begin{cases} -\lambda \alpha t + x, & x < 0 \\ -\lambda \beta t + x, & x \ge 0 \end{cases}$$

so the characteristic lines starting from points equidistant from $x = 0$ do not intersect if the initial surface is convex, which in this case means $\alpha > \beta$. In this case $\rho(x,t)$ is (i.e. solutions to equation 9 with $D = 0$)

$$\rho(x,t) = \begin{cases} -\alpha, & x < -\lambda\alpha t \\ \frac{x}{\lambda t}, & -\lambda\alpha t < x < -\lambda\beta t \\ -\beta, & x > -\lambda\beta t \end{cases}$$

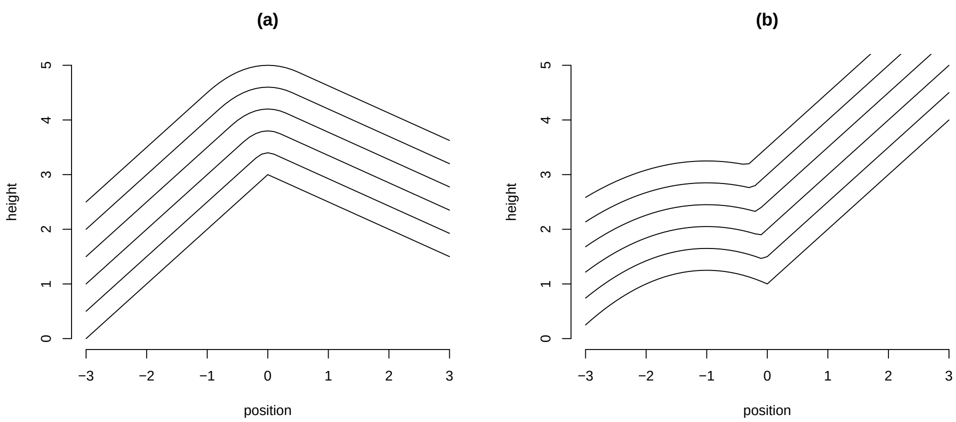
and $h(x,t)$ can be found by integrating equation (11) in time, given $\rho(x,t)$ and $h(x,0)$

$$h(x,t) = \begin{cases} \alpha x + \frac{\lambda}{2}\alpha^2 t + (\lambda + v)t + h_0, & x < -\lambda\alpha t \\ -x^2/2\lambda t + (\lambda + v)t + h_0, & -\lambda\alpha t < x < -\lambda\beta t \\ \beta x + \frac{\lambda}{2}\beta^2 t + (\lambda + v)t + h_0, & x > -\lambda\beta t \end{cases}$$

This case is illustrated in Figure 4(a).

Figure 4: Non-diffusive solutions to the KPZ Equation

The following shows the surface growth as described by equation (11) for $D = 0$, $\lambda = 1$, and $v = 1$. In (a) the initial height is described by two linear segments, as described in the text, with parameters $\alpha = 1$ and $\beta = -0.5$. In (b) the initial height is described by a linear and parabolic segment, with parameters $a = 1$, $\alpha = 0.5$ and $\beta = 1$. In each case the figure shows the surface height in steps of 0.2 time units, from $t = 0$.



Concave surface, no diffusion If the surface is concave, the characteristic lines will intersect, so the solution is different. Consider the initial surface described by a linear segment and a parabolic segment, e.g. for some constant a

$$h(x,0) = \begin{cases} h_0 + \alpha a^2 - \alpha(x+a)^2, & x \le 0 \\ h_0 + \beta x, & x \ge 0 \end{cases}$$

then $\rho(x,t) = -\partial h/\partial x$ has initial condition

$$\rho(x,0) = \begin{cases} 2\alpha(x+a), & x < 0 \\ -\beta, & x > 0 \end{cases}$$

and the characteristic lines are

$$x(t) = \begin{cases} 2\lambda\alpha(x+a)t + x, & x < 0 \\ -\lambda\beta t + x, & x \ge 0 \end{cases}$$

In this case, if $2\alpha a + \beta > 0$ the characteristic lines starting equidistant from the origin intersect when

$$|x| = x^*(t) = \frac{\lambda t}{2}\left(\frac{2\alpha a + \beta}{1 + \lambda\alpha t}\right)$$

so we can write the initial condition for $\rho(x,t) = -\partial h/\partial x$ on either side of $x^*(t)$ as

$$\rho(x,0) = \begin{cases} 2\alpha(\frac{x+a}{1+2\lambda\alpha t}), & x < x^* \\ -\beta, & x > x^* \end{cases}$$

and then again integrating once we have

$$h(x,t) = \begin{cases} -\alpha\frac{(x+a)^2}{1+2\lambda\alpha t} + \alpha a^2 + (\lambda+v)t + h_0, & x < x^* \\ \beta x + \frac{\lambda}{2}\beta^2 t + (\lambda+v)t + h_0, & x \geq x^* \end{cases}$$

where

$$x^*(t) = -\frac{1}{2\alpha}\left(2\alpha a + \beta(1+2\lambda\alpha t) - \sqrt{1+2\lambda\alpha t}(2\alpha a+\beta)\right)$$

This case is illustrated in Figure 4(b).

KPZ Equation: drift and diffusion

There is a solution to equation (11) consisting of parabolic segments like those considered in the examples, e.g.

$$h(x,t) = A + (\lambda+v)t - \frac{D}{\lambda}\ln(2\lambda t - B) - \frac{(x-x_0)^2}{2\lambda t - B}$$

which can be verified directly. As an alternative, we can use a transformation of the height $w(x,t) = e^{\lambda/2D}h(x,t)$ with separation of variables to get the solution

$$h(x,t) = \frac{2D}{\lambda}\ln\left(\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-y)^2}{4Dt}+\frac{\lambda}{2D}h_0(y)}}{\sqrt{4\pi Dt}}dy\right) + (\lambda+v)t \tag{12}$$

This examples in this section on the KPZ equation follows the presentation given in ([1]).

Appendix: Method of Characteristics

For a quick introduction to the method of characteristics, see (wikipedia `http://en.wikipedia.org/wiki/Method_of_characteristics`). The general idea is to take a first order PDE and reduce it to two first order ODEs, as follows.

Consider the a quasi-linear PDE with the general form (equation (3) for example)

$$A(g,x,t)\frac{\partial g}{\partial t} + B(g,x,t)\frac{\partial g}{\partial x} = C(g,x,t) \tag{13}$$

where the initial condition at $t = 0$ is $g(x, 0) = f(x)$. Equation (13) is of the form

$$(A, B, C) \cdot (g_x, g_t, -1) = 0 \tag{14}$$

and since the vector $(g_x, g_t, -1)$ is normal to the solution surface $g(x, t) = z$, then by virtue of equation (14) the vector (A, B, C) is tangent to the solution surface. Thus for any point satisfying the initial condition, we can follow the vector (A, B, C) along the solution surface. The curves generated by following (A, B, C) are the characteristic curves.

The characteristic curves are represented parametrically, one parameter indicating the starting position (per the initial condition) and the other marking how far along the characteristic curve we are. In this parametric representation

$$x = x(r; s) \quad t = t(r; s) \quad g = g(r; s)$$

By construction, the tangent to the characteristic curve (i.e. (x_r, t_r, g_r)) is the tangent to the solution (i.e. the vector (A, B, C)), so

$$x_r = A(g, x, t) \quad t_r = B(g, x, t) \quad g_r = C(g, x, t)$$

Since only derivatives in r appear, the solution to the PDE (equation (13)) has been reduced to the solution to a system of ODEs, with initial conditions

$$s = x(0; s) \quad 0 = t(0; s) \quad f(s) = g(0; s)$$

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