

Lecture 23: Continuous-Time Random Walks

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In previous lectures, we have restricted ourselves to studying random walks where the waiting time between steps is a constant, discrete amount. In this lecture, we will relax this assumption, and consider the case when the times between steps can follow a continuous distribution. Such cases can often lead to anomalous diffusion, where the width $L(t)$ of the probability density function of a walker $X(t)$ scales like t^ν , where ν is no longer $\frac{1}{2}$ as would be expected from the Central Limit Theorem, so we begin with a general discussion of when this behavior can arise.

1 Anomalous diffusion

Consider first when $\nu > \frac{1}{2}$; in that case, we say that a random walker is *superdiffusive*. The simplest model of a superdiffusive process is a Lévy flight: as discussed in lecture 22, this is a random walk with IID steps where $\sigma^2 = \infty$. For example, suppose $p(x) \sim A|x|^{-1-\alpha}$ as $|x| \rightarrow \infty$, where $0 < \alpha < 2$. Then if

$$P_N(x) = \frac{1}{N^\nu} \phi\left(\frac{x}{N^\nu}\right),$$

we have $\phi_N(z) \rightarrow l_{\alpha,0}(z) \propto |z|^{-1-\alpha}$ as $|z| \rightarrow \infty$. For the case of $\alpha = 2$ we still get a gaussian distribution but the scaling goes like $\sqrt{N \log N}$, which is a case of the strong Central Limit Theorem.

Lévy flights have a number of important applications, such as in modeling financial time series, the albatross migration, or polymers (as described in problem set 4). They can also be used to describe Brownian motion in gases in cases of high Knudsen number, where the mean free path of a particle is large compared to the system size. Consider the case of a gas particle in a channel between two parallel plates a distance L apart, and suppose that each time the particle hits the wall, it bounces off at a random angle θ , chosen uniformly over the range $0 < \theta < \pi$. In the absence of any interactions, the horizontal displacement of the particle before it hits the opposing wall is $x = L \cot \theta$. We see that the probability density function of x is given by

$$\begin{aligned} p(x)dx &= p(\theta)d\theta \\ p(x)dx &= \frac{d\theta}{\pi} \\ p(x)dx &= \frac{Ldx}{L^2 + x^2} \end{aligned}$$

and thus follows a Cauchy Distribution, $l_{1,0}(L, x)$. Hence after N steps have taken place, the horizontal displacement of the particle follows a Cauchy Distribution $l_{1,0}(NL, x)$. Note however that this model does not take into account the fact that longer steps will invariably take longer. A

more appropriate model may be the Lévy walk, where the time to take a given step is taken into account, perhaps by assuming that particle moves at constant velocity.

If $\nu < \frac{1}{2}$ then we call a process *subdiffusive*; such processes can often occur when we consider diffusion in a disordered media, when a walker's path may frequently be blocked. An example is DNA electrophoresis in a gel: a strand of DNA is often tangled and trapped in the long chains that make up the gel. Subdiffusion can also occur for processes with long trapping times, where the expected wait between steps is infinite.

2 Random waiting times

To consider a continuous time random walk, we must first develop a mathematical framework for handling random waiting times between steps, and since these times must be positive, it is natural to make use of the Laplace transform. Let τ be a random variable for the waiting time, with probability density function $\psi(t)$. Let $N(t)$ be the random variable for the number of events up to time t with probability distribution $\mathcal{P}(N, t)$, and define

$$\Psi(t) = \int_t^\infty \psi(s) ds$$

to be the probability that an event does not take place before time t . We see that

$$\mathcal{P}(1, t) = \int_0^t \psi(s) \Psi(t-s) ds = (\psi * \Psi)(t)$$

and $\mathcal{P}(N, t) = (\psi^{*N} * \Psi)(t)$. Taking the Laplace transform gives

$$\tilde{\mathcal{P}}(N, s) = \tilde{\psi}(s)^N \tilde{\Psi}(s) = \tilde{\psi}(s)^N \frac{1 - \tilde{\psi}(s)}{s}.$$

Depending on whether the expected waiting time is infinite, we have two different forms for the Taylor expansion of this expression as $s \rightarrow 0$. If $\bar{\tau} < \infty$, then

$$\tilde{\mathcal{P}}(N, s) \sim 1 - \bar{\tau}s + \frac{\langle \tau^2 \rangle}{2} s^2 - \dots$$

from which we can calculate moments of $N(t)$. Alternatively, if $\bar{\tau} = \infty$, then

$$\tilde{\mathcal{P}}(N, s) \sim 1 - As^\gamma$$

where $0 < \gamma < 1$. In this case, we can make use of analogs for the Laplace Transform of the Tauberian Theorems, originally discussed in lecture 6. For $s \rightarrow 0$ and $t \rightarrow \infty$ we have

$$\tilde{f}(s) \sim 1 - As^\gamma \Leftrightarrow f(t) \sim Bt^{-1-\gamma}$$

for $0 < \gamma < 1$, and

$$\tilde{f}(s) \sim As^{-\beta} \Leftrightarrow f(t) \sim \frac{At^{1+\beta}}{\Gamma(\beta)}$$

for $\beta > 0$.

An example of the above for a finite $\bar{\tau}$ is where the waiting times are exponentially distributed according to $\psi(t) = \lambda e^{-\lambda t}$. The Laplace transform of $\psi(t)$ is given by

$$\tilde{\psi}(s) = \frac{\lambda}{s + \lambda}$$

and hence

$$\tilde{\mathcal{P}}(N, s) = \left(\frac{\lambda}{s + \lambda}\right)^N \left(1 - \frac{\lambda}{s + \lambda}\right) \frac{1}{s} = \frac{\lambda^N}{(s + \lambda)^{N+1}}.$$

This can be inverted; the only contribution to the integral comes from the pole of order $N + 1$ at $s = -\lambda$ and hence

$$\mathcal{P}(N, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \tilde{\mathcal{P}}(N, s) ds = \text{Res}(e^{st} \tilde{\mathcal{P}}, s = -\lambda) = \lambda^N \left. \frac{d^N}{ds^N} e^{st} \right|_{s=-\lambda} = \frac{(\lambda t)^N e^{-\lambda t}}{N!}$$

which is a Poisson distribution with mean $\lambda t = t/\bar{\tau}$. For an example when $\bar{\tau}$ is infinite, consider the case when $\psi(t) \sim At^{-1-\gamma}$ as $t \rightarrow \infty$, and $0 < \gamma < 1$. Using the Tauberian Theorem, we know that as $s \rightarrow 0$,

$$\tilde{\psi}(s) \sim 1 - Bs^\gamma.$$

To obtain the scaling behavior for the expected number of steps, $\bar{N}(t)$, we make use of

$$\bar{N}(t) = \sum_{n=0}^{\infty} N \mathcal{P}(N, s)$$

from which we obtain

$$\begin{aligned} \tilde{\bar{N}}(s) &= \sum_{N=0}^{\infty} \tilde{\mathcal{P}}(N, s) N \\ &= \left(\frac{1 - \tilde{\psi}(s)}{s}\right) \sum_{N=0}^{\infty} N \tilde{\psi}(s)^N \\ &= \frac{\tilde{\psi}(s)}{s(1 - \tilde{\psi}(s))}. \end{aligned}$$

Since $\tilde{\psi}(s) \sim 1 - Bs^\gamma$, we know that $\tilde{\bar{N}}(s) \sim s^{-1-\gamma}/B$. Hence $\bar{N}(t) \propto t^\gamma$, which scales slower than t ; this behavior is often referred to as “fractal” since there is no characteristic time scale. For a finite mean, we would have $\tilde{\psi}(s) \sim 1 - \bar{\tau}(s)$ and hence $\tilde{\bar{N}}(s) \sim (\bar{\tau}s^2)^{-1}$ giving $\bar{N}(t) \propto t/\bar{\tau}$, which does have a characteristic time scale.

3 Continuous Time Random Walk

We now consider the case of a random walker which takes IID displacements from distribution $p(x)$, that occur following a waiting time distribution $\psi(t)$. Here we consider the case of a separable continuous-time random walk, originally discussed by E. Montroll and G. H. Weiss (1965), where the waiting times and the displacements are independent. Such a model is often inappropriate for superdiffusion, since long steps are often correlated with longer times, but it can be accurate for

many subdiffusive processes, since a long waiting time is often dominant. Let $P(x, t)$ be the PDF for the position X at time t , so

$$X(t) = \sum_{n=1}^{N(t)} x_n$$

which is a random sum of random variables. From Bachelier's equation we have

$$P(x, t) = \sum_{N=0}^{\infty} \mathcal{P}(N, t) p(x)^{*N},$$

and taking a Fourier transform in space and a Laplace transform in time gives

$$\tilde{P}(k, s) = \sum_{N=0}^{\infty} \tilde{\mathcal{P}}(N, s) \hat{p}(k)^N = \sum_{N=0}^{\infty} \frac{1 - \tilde{\psi}(s)}{s} \tilde{\mathcal{P}}(s)^N \hat{p}(k)^N = \frac{1 - \tilde{\psi}(s)}{s(1 - \tilde{\psi}(s)\hat{p}(k))}$$

using the fact that $|\tilde{\psi}\hat{p}| < 1$ for a valid PDF. This is the Montroll-Weiss equation, and it will be discussed in more detail in later lectures.