# Lecture 22: Lévy Distributions

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## 1 Central Limit Theorem (from previously)

Let $\{x_i\}$ be IID random variables with finite mean $\mu$ and finite variance $\sigma^2$. Then the RV

$$S_n = \frac{1}{\sigma\sqrt{n}}\left(\sum_{j=1}^{n} x_j - n\mu\right) \tag{1}$$

converges to a standard $N(0,1)$ Gaussian density.

## 2 Introduction

This lectures covers similar material to Lectures 12 and 13 in the 2003 lecture notes. Also, see Hughes[1] §4.3.

Motivating question: What are the limiting distributions for a sum of IID variables? (And, in particular, those who violate the assumptions of the CLT.)

## 3 Lévy stability laws

Definitions: $\{x_i\}$ RVs and $X_N = \sum_{n=1}^{N} x_n$

Characteristic function:

$$\hat{p}_n(k) = [\hat{p}(k)]^n \tag{2}$$

$$\hat{p}_{n\times m}(k) = [\hat{p}_m(k)]^n = [\hat{p}(k)]^{n\times m} \tag{3}$$

Introduce new variables $Z_N = \frac{X_N}{a_N}$. The PDF of the $Z_N$'s is given by $F_N(x) = p_N \frac{x}{a_N}\frac{1}{a_N}$.

$$\hat{F}_N(a_N k) = \hat{p}_N(k) \tag{4}$$

Substituting in,

$$\hat{F}_{n\times m}(a_{nm}k) = \left[\hat{F}_m(a_m k)\right]^n \tag{5}$$

[1] B. Hughes, *Random Walks and Random Environments*, Vol. I, Sec. 2.1 (Oxford, 1995).

As $n \to \infty$, we want $\hat{F}_n(k)$ to tend to some fixed distribution $\hat{F}(k)$. Say that $\lim_{m\to\infty} \frac{a_{nm}}{a_m} = c_n$,

$$\hat{F}(kc_n) = \left[\hat{F}(k)\right]^n \tag{6}$$

Suppose I want to find

$$\hat{F}(k\mu(\lambda)) = \left[\hat{F}(k)\right]^\lambda \tag{7}$$

Introduce $\psi(k) = \log \hat{F}(k)$

$$\psi(k\mu(\lambda)) = \lambda\psi(k) \tag{8}$$

Differentiate:

$$\begin{aligned}
k\mu'(\lambda)\,\psi'(k\mu(\lambda)) &= \psi(\lambda) && (9)\\
\mu(\lambda = 1) &= 1 && (10)\\
k\mu'(1)\,\psi'(k) &= \psi(k) && (11)\\
\frac{d\psi}{dk} &= \frac{\psi}{\mu'(1)\,k} && (12)
\end{aligned}$$

This has a solution:

$$\psi(k) = \begin{cases} v_1|k|^\alpha & k > 0 \\ v_2|k|^\alpha & k < 0 \end{cases} \tag{13}$$

$$\hat{F}(k) = \begin{cases} \exp(v_1|k|^\alpha) & k > 0 \\ \exp(v_2|k|^\alpha) & k < 0 \end{cases} \tag{14}$$

$$v_1, v_2 \in \mathcal{C} \tag{15}$$

Thus we have this condition:

$$\begin{aligned}
\hat{F}(-k) &= \hat{F}(k)^* && (16)\\
&\Longrightarrow \hat{F}(k) = \exp\left((c_1 + ic_2 sgn(k))\,|k|^\alpha\right) && (17)\\
&c_1, c_2 \in \Re && (18)
\end{aligned}$$

We can also write:

$$\hat{F}(k) = \exp\left[-a|k|^\alpha\left(1 - i\beta\tan\frac{\alpha\pi}{2} sgn(k)\right)\right] \tag{19}$$

In this equation, $\beta$ relates to the skewness of the distribution.

# 4 Lévy distribution

For above $\hat{F}$ when $\beta = 0$ (ie. symmetric limiting distribution), you get the Lévy distribution. It is defined in terms of its Fourier transform.

$$\hat{L}_\alpha(a, k) = \hat{F}(k) = \exp(-a|k|^\alpha) \tag{20}$$

Two special cases of the the Lévy distribution are $\alpha = 1, 2$.

| | | | |
|---|---|---|---|
| $\alpha = 2$ | $\hat{F}(k) = \exp\left(-ak^2\right)$ | like Gaussian but more general | (21) |
| $\alpha = 1$ | $\hat{F}(k) = \exp(-a|k|)$ | Cauchy | (22) |

### 4.1 Large x expansions of the Lévy distribution

$$L_\alpha(a,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp ikx - a|k|^\alpha \mathrm{d}k \tag{23}$$

$$= \frac{1}{\pi}\int_{0}^{\infty} \cos kx \exp -ak^\alpha \mathrm{d}k \tag{24}$$

Integrate by parts: (25)

$$= \frac{a\alpha}{\pi}\int_{0}^{\infty} \sin \frac{k|x|}{|x|} \exp -ak^\alpha k^{\alpha-1}\mathrm{d}k \quad = \frac{a\alpha}{\pi|x|^{1+\alpha}}\int_{0}^{\infty} \xi^{\alpha-1}\sin\xi \exp \frac{-a\xi^\alpha}{|x|^\alpha}\mathrm{d}\xi \tag{26}$$

Where we have made the following substitutions:

$$\xi = k|x|, \mathrm{d}\xi = \mathrm{d}k|x| \tag{27}$$

Taking the limit $x \to \infty$:

$$\sim_{x\to\infty} \quad \frac{a\alpha}{\pi|x|^{1+\alpha}}\int_{0}^{\infty} \xi^{\alpha-1}\sin\xi \mathrm{d}\xi \tag{28}$$

$$\sim \quad \frac{a\alpha\Gamma(\alpha)\sin\frac{\pi\alpha}{2}}{\pi|x|^{1+\alpha}} \qquad \text{power-law tail} \tag{29}$$

### 4.2 Small x expansion

$$L_\alpha(a,x) = \frac{1}{\pi}\int_{0}^{\infty} \cos kx \exp -ak^\alpha \mathrm{d}k \tag{30}$$

$$\text{Taylor expand cosine:} \quad = \frac{1}{\pi}\int_{0}^{\infty}\sum_{n=0}^{\infty} (-1)^n \frac{(kx)^{2n}}{(2n)!} \exp -ak^\alpha \mathrm{d}k \tag{31}$$

Substitute: $w = ak^\alpha \quad dw = \alpha a k^{\alpha-1}\mathrm{d}k$ (32)

$$= \frac{1}{\pi}\sum_{n=0}^{\infty}\int_{0}^{\infty} (-1)^n \frac{(x)^{2n}}{(2n)!} \frac{w}{a}^{2n/\alpha} \exp -w \frac{\mathrm{d}w}{\alpha a \frac{w}{a}^{\frac{\alpha-1}{\alpha}}} \tag{33}$$

We're only interested in evaluating this equation as $x \to 0$:

$$= \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\frac{(x)^{2n}}{a^{\frac{2n+1}{\alpha}}}\Gamma\left(\frac{2n+1}{\alpha}\right) \tag{34}$$

$$\sim \frac{1}{\pi}\frac{\Gamma\left(\frac{1}{\alpha}\right)}{\alpha a^{a/\alpha}} \quad \text{increases as } \alpha \to 0 \tag{35}$$

## 5 Analogy with Central Limit Theorem?

### 5.1 Gnedenko-Doblin Theorem

$\frac{1}{a_N} p_N \frac{x}{a_N} \to L_{\alpha,\beta}(a,x)$ *iff* the CDF $P(x)$ satisfies

1. $\lim\limits_{x\to\infty} \frac{p(-x)}{a-p(x)} = \frac{1-\beta}{1+\beta}$

   This is akin to the "amount" of probability on one side of the tail.

2. $\lim\limits_{x\to\infty} \frac{1-p(x)+p(-x)}{1-p(rx)+p(-rx)} = r^{\alpha} \quad \forall r$

   We can think of this as saying how fast it is decaying.

## 5.2 Theorem

The distribution $p(x)$ is in the basin of attraction of $L_{\alpha,\beta}(a,x)$ if

$$p(x) \sim \frac{A_{\pm}}{|x|^{1+\alpha}} \quad x \to \pm\infty \tag{36}$$

$$\beta = \frac{A_+ - A_-}{A_+ + A_-} \tag{37}$$

If $p(x)$ has power-law tails, then the sum of RVs goes to a Lévy distribution.

## 5.3 Scaling and superdiffusivity

Suppose that $\{x_i\}$ follow a $L_{\alpha}(a,x)$ distribution, then $\hat{p}(k) = \exp -a|k|^{\alpha}$ and then the characteristic function at $x_N = \sum\limits_{n=1}^{N} x_n$ is given by

$$\hat{p}_N(k) = [\exp -a|k|^{\alpha}]^N = \exp -a|k|^{\alpha} N \tag{38}$$

$$p_N(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \exp ikx - aN|k|^{\alpha}\mathrm{d}k \tag{39}$$

Now we substitute:

$$k = VN^{-1/\alpha}, \mathrm{d}k = -\mathrm{d}VN^{-1/\alpha} \tag{40}$$

$$p_N(k) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \exp iV\left(\frac{x}{N^{1/\alpha}}\right) - a|V|^{\alpha}\frac{\mathrm{d}V}{N^{1/\alpha}} = \frac{1}{N^{1/\alpha}} L_{\alpha}\left(a, \frac{x}{N^{1/\alpha}}\right) \tag{41}$$

Thus, the width scales like $N^{1/\alpha}$.

If $\alpha > 2$, we get scaling that exceeds "square-root" scaling, ie. a superdiffusive process.