Lecture 17: Return and First Passage on a Lattice[*]

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1   Return Probability on the Integer Lattice ($d = 1$)

To begin, we consider a basic example of a discrete first passage process. Consider an unbiased Bernoulli walk on the integers starting at the origin. We wish to determine the probability, R, that the walker eventually returns to 0, regardless of the number of steps it takes.

Without loss of generality, say the walker's first step is to 1. From here, the walker can either step left to 0 or can return to 1 n times before stepping back to 0. To avoid double counting, we must also assert that if the walker comes back to 1 n times, it cannot step left to 0 any of those times except the last. Since half of the possible paths from 1 back to 1 stay to the right of 1, the probability of returning to 1 exactly once before hitting 0 is $R/2$. Likewise, the probability of returning to 1 n times before stepping back to 0 is $(R/2)^n$. Thus:

$$R \; = \; \frac{1}{2} + \left[\sum_{n=1}^{\infty}(R/2)^n\right] * \frac{1}{2} \; = \; \left[\sum_{n=0}^{\infty}(R/2)^n\right] * \frac{1}{2} \; = \; \frac{1}{2(1 - R/2)} \; = \; \frac{1}{2 - R}.$$

This means,

$$R^2 - 2R + 1 = 0 \implies (R - 1)^2 = 0$$

and since probabilities cannot be negative, this means $R = 1$.

In essence, we've just deduced that a drunk man who aimlessly wanders away from a pub will eventually return, as long as he is confined to a long narrow street. Based on the continuum analysis of the previous lecture, however, he will probably not make it back the same day (or the same week), since the expected return time is infinite!

The preceding simple analysis is not easily generalized to higher dimensions, where the situation can be quite different, e.g. if the drunk man wanders about in a two-dimensional field. As the dimension increases, it makes sense that the walker is less likely to ever find by chance the special point where he started. In the next lecture, we will address the effect of dimension in the return problem (Polyá's theorem), but first we will develop a "transform" formalism for discrete random walks on a lattice, analogous to the Fourier and Laplace transform methods used earlier for continuous displacements. Of course, we could use the same continuum formulation with generalized functions (like $\delta(x)$) to enforce lattice constraints, but it is simpler to work with discrete "generating functions" right from the start.

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2   Generating Functions on the Integers

Rather than keeping track of a sequence of discrete probabilities, $\{P(X = n)\}$, it is convenient to encode the same information in the expansion coefficients, $P_n = P(X + n)$, of the "probability generating function" (PGF), defined as follows,

$$f(\xi) = \sum_{n=n_0}^{\infty} P_n \xi^n$$

There are two common cases:

1. For probabilities defined on all integers, $n_0 = -\infty$, the PGF is the analytic continuation of a Laurent series,

   $$f(z) = \sum_{-\infty}^{\infty} P_n z^n$$

   which converges in some annulus in the complex plane, $R_1 < |z| < R_2$. The probabilities are recovered from the PGF by a contour integral around this annulus once counter-clockwise,

   $$P_n = \frac{1}{2\pi i} \oint \frac{f(z)dz}{z^{n+1}}$$

   When evaluated on the unit circle, $z = e^{-i\theta}$, the Laurent series redices to a complex Fourier series,

   $$f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} P_n e^{in\theta}$$

   which is the discrete analog of our previous Fourier transform in "space".

2. For probabilities defined on the non-negative integers, $n_0 = 0$, the PGF is the analytic continuation of a Taylor series,

   $$f(z) = \sum_{0}^{\infty} P_n z^n$$

   which converges in a disk in the complex plane, $|z| < R$. When evaluated on the real axis inside the unit disk with $z = e^{-s}$ ($s > 0$), the Taylor series ressembles a "discrete Laplace transform",

   $$f(e^{-s}) = \sum_{n=0}^{\infty} P_n e^{-sn}$$

   which is the discrete analog of our previous Laplace transform in "time".

For an intuitive discussion of generating functions and their relation to Fourier and Laplace transforms in the present context, see Redner's book, *A Guide to First Passage Processes* (recommended reading).

For the remainder of this lecture, we will focus on case 2, which has relevance for the return problem in one dimension. In this case, similar to the continuous transforms, the PGF has some very useful properties:

- $f(1) = \sum_{n=0}^{\infty} P(X = n) = 1$ (by normalization), which implies that the PGF converges inside the unit disk, $|\xi| < 1$.

- All of the moments of the distribution are encoded in Taylor expansion of the PGF as $\xi \to 1^-$ (analogous to the Taylor coefficients of the Fourier at the origin). For example: $f'(\xi) = \sum_{n=0}^{\infty} P(X = n)n\xi^{n-1} \implies f'(1) = \sum_{n=0}^{\infty} nP(X = n) = \langle X \rangle$. Similarly, $f''(\xi) = \sum_{n=0}^{\infty} n(n-1)\xi^{n-2}P(X = n) = \langle X^2 \rangle - \langle X \rangle$ implying that $\langle X^2 \rangle = f''(1) + f'(1)$.

- A "discrete convolution theorem" also holds. Suppose Y has probability generating function $g(\xi)$. Then the PGF of $Z = X + Y$ is
$h(\xi) = \sum_{n=0}^{\infty} P(X + Y = n)\xi^n = \sum_{n=0}^{\infty} \sum_{i=0}^{n} P(X = i)P(Y = n - i)\xi^n$. Letting $k = n - i$ we have $h(\xi) = \sum_{i=0}^{\infty} P(X = i) \sum_{k=0}^{\infty} P(Y = k)\xi^{i+k} = f(\xi)g(\xi)$.

For example, consider a Poisson distribution with parameter λ:

$$P(X = n) = e^{-\lambda}\lambda^n/n!$$

$$\implies f(\xi) = \sum_{n=0}^{\infty} e^{-\lambda}(\lambda\xi)^n/n! = e^{-\lambda}e^{\lambda\xi} = e^{\lambda(\xi-1)}.$$

We get $f(1) = e^0 = 1$ as we expect and $f'(1) = \lambda = \langle X \rangle$. If we let Y be Poisson with parameter μ, we may define the variable $Z = X + Y$. By the last property we get that the PGF for Z is $h(\xi) = e^{(\lambda+\mu)(\xi-1)}$ thus telling us that the sum of two Poisson variables also has a Poisson distribution (with parameter $\lambda + \mu$).

3 First Passage on a Lattice

Define $P_n(s|s_0)$ as the probability of being at lattice point s after n steps given that the walk started at s_0. Also, define $F_n(s|s_0)$ as the probability of arriving at site s for the first time on the nth step, given that the walker begins at s_0.

One condition we know must hold is

$$\sum_s P_n(s|s_0) = 1$$

since the walker must be somewhere on the lattice. We may also define

$$R(s|s_0) = \sum_{n=1}^{\infty} F_n(s|s_0)$$

as the probability that site s is ever reached by a walker strating from site s_0. Our initial conditions for a walker beginning at s_0 are
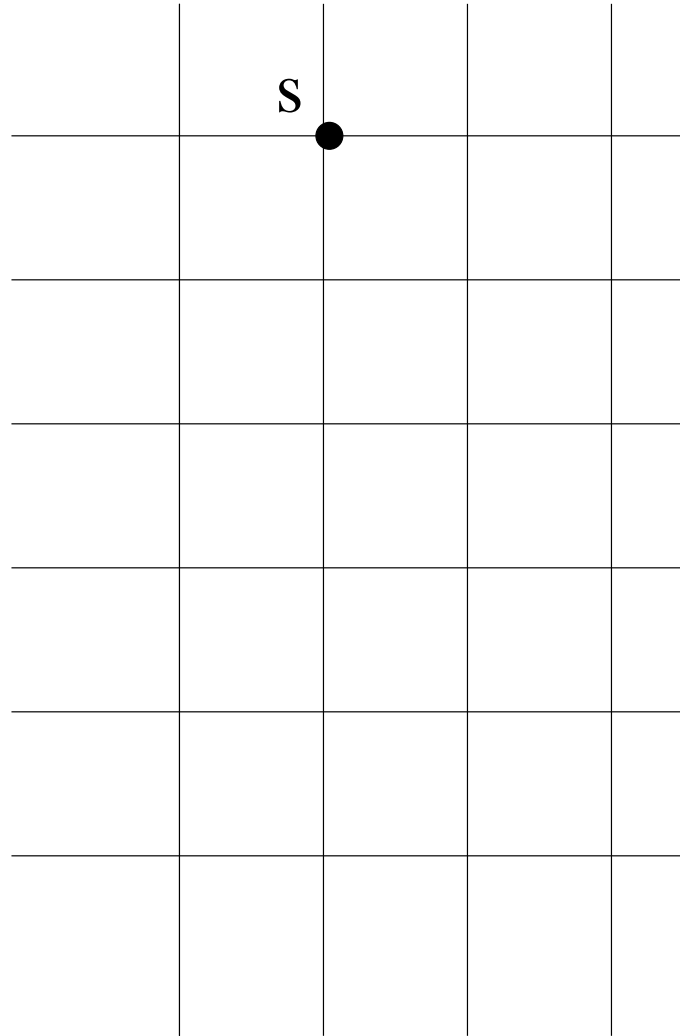
$$P_0(s|s_0) = \delta_{ss_0} \quad \text{and} \quad F_0(s|s_0) = 0$$

where δ is the Kroenecker delta function. Use the following notation for the generating functions:

$$P(s|s_0; \xi) = \sum_{n=0}^{\infty} P_n(s|s_0)\xi^n$$

$$F(s|s_0; \xi) = \sum_{n=0}^{\infty} F_n(s|s_0)\xi^n.$$

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The odds of a walker reaching s from s_0 are the same as the odds of a walker first arriving at s in j steps and then returning to s in $n - j$ steps. Thus we can write:

$$P_n(s|s_0) = \sum_{j=1}^{n} F_j(s|s_0) P_{n-j}(s|s)$$

as long as $n \geq 1$. In the case that $n = 0$, $P_0(s|s_0) = \delta_{ss_0}$ so altogether we have

$$P_n(s|s_0) = \delta_{ss_0} \delta_{n0} + \sum_{j=1}^{n} F_j(s|s_0) P_{n-j}(s|s)$$

By the convolution-like property of PGF's we can now write

$$P(s, s_0; \xi) = \delta_{ss_0} + F(s|s_0; \xi) P(s|s; \xi)$$

$$\implies F(s|s_0; \xi) = \frac{P(s|s_0; \xi) - \delta_{ss_0}}{P(s|s; \xi)}.$$

This is comparable to the result in the continuum case using a Laplace Transform. Using this, we can write

$$R(s|s_0) \;=\; \sum_{n=1}^{\infty} F_n(s|s_0) \;=\; F(s|s_0; 1) \;=\; \frac{P(s|s_0; 1) - \delta_{ss_0}}{P(s|s; 1)}$$

and thus the probability of return is

$$\boxed{R(s_0|s_0) \;=\; 1 - \frac{1}{P(s|s_0; 1)}} \;.$$

4   First Passage Example

Let us now consider a biased Bernoulli walk on the integers.

$$P_{2m}(0|0) \;=\; \binom{2m}{m} p^m q^m$$

where we use $2m$ because a walk that returns must do so in an even number of steps.

$$P(0|0; \xi) \;=\; \sum_{m=0}^{\infty} \binom{2m}{m} p^m q^m \xi^{2m}$$

$$=\; (1 - 4pq\xi^2)^{-1/2}$$

and hence the probability of return is

$$R(0|0) \;=\; 1 - \sqrt{1 - 4pq}$$
$$=\; 1 - \sqrt{1 - 4p(1 - p)}$$
$$=\; 1 - \sqrt{(2p - 1)^2}$$
$$=\; 1 - |2p - 1|$$

This result agrees with our first example, just let $p = 1/2$ and we get $R = 1$.

5   Reflection Principle

The following is the beginning of a derivation of the "arc-sine law", given as a simulation problem on the first problem set (fraction of the time spent in a given region).

Consider a symmetric Bernoulli walk on the integers. Let $N(x, t)$ be the number of paths from $(0, 0)$ to (x, t). We know that

$$N(x, t) \;=\; \frac{t!}{(\frac{t+x}{2})! \, (\frac{t-x}{2})!} \;=\; \binom{t}{\frac{t+x}{2}}$$

Let $X_y(x, t)$ be the number of paths which cross $y > x$. We may create a new path, one which begins at the origin and ends at $2y - x$ at time t by simply reflecting the last crossing of a path in $X_y(x, t)$ about the line $x = y$.

We have just, in effect, defined a bijection between N and X_y under

$$X_y(x, t) \;=\; N(2y - x, t).$$

This principle has important applications related to first passage and return on a lattice. See for example, Feller, Vol 1 (1970) from the recommended reading for the application to the arc-sine law.

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