# Lecture 16: First Passage in the Continuum Limit

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**Abstract**

We introduce the notion of first passage and talk briefly about the areas of science where it arises naturally. We then take a closer look at the first passage problem in continuous -as opposed to discrete, which will be coming in the next lecture- random processes. Namely we solve the one-dimensional first passage problem, and formulate the general case where the random walker lives in higher dimensions, in terms of a partial differential equation with appropriate initial and boundary conditions.

## 1 Introduction

Consider a sleep-walker walking on the roof of her house who started from a point which we call the origin. One may be interested in the typical time that she will fall down onto the ground from the roof. In the simple case of one dimensional roof,

we could model this problem by a random walker starting at the origin, which represents the starting point of the sleep walker. In this case, because she will have died and not be walking after falling down, we would only be interested in the number of steps after which the night-walker (or the random walker) reaches a fixed position $x$, or the edge of the roof, for the first time. This

first-passage time is denoted by $T(x)$ and is associated with a density $f(x,t)$, which is called the first passage time density.

Some problems of interest are

- Given the probability density function $p(x,t)$ of the random walker, what is $f(x,t)$?
- Will the random walker ever hit $x$? What is the probability that the random walker will ever hit $x$? In other words, what is $\int_0^\infty f(x,t)dt$ ?

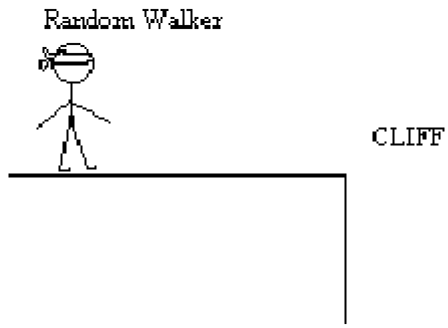


Figure 1: In presence of a cliff, the question is whether or not the walker is going to fall down, and if yes, how long this takes. One is not concerned about what happens after then, often simply because there is no random walk after the fall.

- What is the mean first passage time, i.e. $\langle \tau \rangle = \int_0^\infty t f(x,t) dt = ?$

More generally, one may consider the first passage time from a given point $x_0$, which is not necessarily the origin, to a set $S$ which lives in $d$ dimensions, where $d$ is not necessarily 1. This gives rise to another interesting question: If the walker ever hits $S$, where in $S$ will this hitting happen?

# 2 Applications

Before jumping into the debilitating job of answering all those questions, one would possibly want to make sure that it is worth the work. It turns out that the notion of the first passage time not only enables one to calculate the life-expectancy of a somnambulist on a high roof, but it also has some other useful -although arguably less important- applications in many areas such as statistical mechanics and finance. Hence we now briefly talk about a fraction of the useful applications of this "new" notion, where it makes our insight deeper, our life easier, and sometimes us richer.

## 2.1 Statistical Mechanics

### 2.1.1 Kramers Escape Problem

Kramers Escape problem, which models many physical and chemical processes such as the one-directional transition of a molecule between its different stable states can be handled by first-passage time approach more accurately.

#### 2.1.2 Ehrenfest's Problem

As discussed in the lecture 14, a simple model to resolve the apparent paradoxes between classical mechanics, and classical thermodynamics is the Ehrenfest's model, which consists of two bins and $2N$ balls, say the first bin contains $N+n$ balls, and the second $N-n$. Each ball could represent could represent a particle or a quantity of heat energy, separated into two reservoirs in thermal contact. At each turn, corresponding to elapsing of a short time period $\tau$, we randomly pick one ball, and change its location, i.e. transport it from its current bin to the other. Ehrenfest's problem is concerned about the evolution of the number of balls in each bin under this process of replacing balls.

One may be interested in the typical number of steps after which one configuration is reached from another, for example, time it takes to reach from $n=k$ to $n=0$, from $n=0$ to $n=N$, or from $n=k$ to $n=k$ for the first time. In the last case, this corresponds to the recurrence under Poincare map. These can be calculated from a first-passage process point of view,. By calculating the typical transition times, which for the Zarmelo's case[1] turns out to be much longer than the lifetime of the universe, one can resolve the apparent contradiction between the Newtonian mechanics and statical mechanics, see lecture 14, Loschmidt's and Zarmelo's paradoxes.

#### 2.1.3 Smoluchowski's theory of reaction rates

Consider particles of a substance whose co-existence results in a reaction. One may talk about the average distance between particles. One quantity of interest, then, would be the mean collision time of particles, the multiplicative of inverse of which is known as the reaction rate. This, again, is a first-passage time problem.

#### 2.1.4 Absorbing or Reaction Surfaces

Consider two species of particles A and B wandering randomly in a domain, and their collision results in a reaction only in the presence of a catalysis C. The location of the catalysis C is known, say it is a particular surface. The problem of finding the reaction rate, again, can be handled by first-passage time calculations.

[1] i.e. transition from $n=k$ to $n=k$ for large $k$

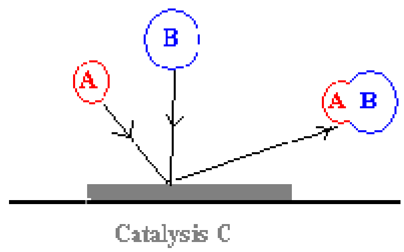


Figure 2: Illustration of a reaction surface

## 2.2 Financial Applications

### 2.2.1 Recovery time

The time it takes for a market to recover from a crash, or to end a bull market, can be modeled as a first-passage time problem. If one believes in the rather unrealistic random walk models for financial markets, this would just be the time it takes for the random walker to return to the central region from extreme regions, for the first time.

### 2.2.2 Pricing/Hedging American Options

Remember the American option, as opposed to its European counterpart, gives its owner the right to sell (or buy- if it is a put option) the underlying at any time during a certain time period, the time until maturity. In this case one is concerned with the time an American option is exercised for the first time-since it cannot be exercised more than once. Therefore pricing and hedging American options is also a first-passage time problem.

# 3 One Dimension: Integral Equation Approach

## 3.1 Solution to the problem in one dimension

Let $p(x,t)$ be the PDF of process $x_t$ starting from $x = 0$ at $t = 0$ being at $x$ at time $t$, and $f(x,t)$ be the PDF of $T(x)$, the first passage time of $x_t$ from 0 to $x$. Then we easily see the relation between $T(x)$ and $f(x,t)$ :

$$\int_{-\infty}^{t} f(x,t')dt' = \Pr(T(x) < t)$$

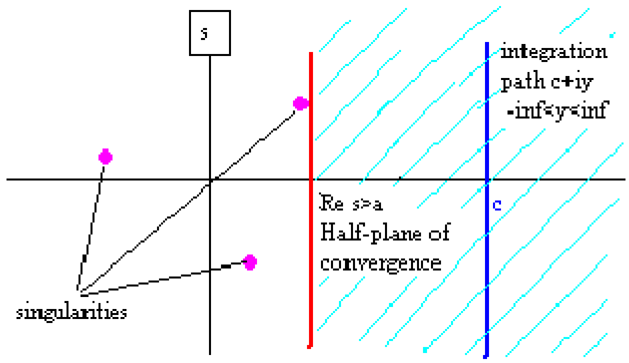


Figure 3: An illustration of the choice of integration path in the formula (2)

We propose the equation

$$p(x,t) = \int_0^t f(x,t')p(0,t-t')dt' \tag{1}$$

This is because

Pr($0 \to x$ at $t$) = Integration over all ways of going $0 \to x$ at $t' < t$ AND returning there ( 0 displacement) in time $t - t'$

The integral in the last formula is just the Laplace convolution of the functions $f(x,t)$ and $p(0,t)$ over the variable $t$.

**Recall:** The convolution of the functions $g$ and $h$ is given by

$$g * h(t) = \int_0^t g(t-t')h(t')dt'$$

and the Laplace transform of a function $g$ is defined by

$$\mathcal{L}[g](s) = \tilde{g}(s) = \int_0^\infty e^{-st}g(t)dt$$

The original function can be recovered by the inverse Laplace transform formula

$$g(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\tilde{g}(s)ds \tag{2}$$

where $c$ is a number selected which has a larger real part than all the singularities of the integrand $\tilde{g}(s)$. Another key fact to keep in mind is that the laplace transform of the convolution is the multiplication of the laplace transforms. In mathematical symbols,

$$\mathcal{L}[g * h](s) = \mathcal{L}[g](s)\mathcal{L}[f](s) = \tilde{g}(s)\tilde{h}(s)$$

We now rewrite the equation (1) in the form of a convolution, i.e.

$$p(x,t) = f(x,t) * p(0,t)$$

where the convolution is taken over the time domain. Taking the Laplace transforms of both sides, i.e. multiplying both sides by $e^{-st}$ and integrating from 0 to $\infty$ over variable $t$, we reach the simple equation

$$\tilde{p}(x,s) = \tilde{f}(x,s)\tilde{p}(0,s) \tag{3}$$

where

$$\tilde{p}(x,s) = \int_0^\infty e^{-st} p(x,t)dt$$

$$\tilde{f}(x,s) = \int_0^\infty e^{-st} f(x,t)dt$$

The algebraic equation (3) can easily be solved for $\tilde{f}(x,s)$ :

$$\tilde{f}(x,s) = \frac{\tilde{p}(x,s)}{\tilde{p}(0,s)} \tag{4}$$

Therefore the PDF $f(x,t)$ is given by the inverse Laplace transform formula

$$f(x,t) = \int_{c-i\infty}^{c+i\infty} e^{st} \frac{\tilde{p}(x,s)}{\tilde{p}(0,s)} ds \tag{5}$$

where $c$ is to the right of the singularities of the integrand in question.

## 3.2 Example: Unbiased Diffusion

Unbiased diffusion is to say that the coefficients are $D_1 = 0$, $D_2 = D$, in which case the Fourier transform of the PDF $p(x,t)$ is given by

$$\hat{p}(k,t) = e^{-k^2 Dt}$$

where $p(x,t)$ can be expressed as

$$p(x,t) = \int_{-\infty}^{\infty} e^{ikx} \hat{p}(k,t) \frac{dk}{2\pi}$$

Then the Laplace transform of $p(x,t)$ can be calculated as

$$\begin{aligned}\tilde{p}(x,s) &= \int_{-\infty}^{\infty} e^{ikx}\left[\int_0^{\infty} e^{-st}e^{-k^2Dt}dt\right]\frac{dk}{2\pi} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ikx}}{s+k^2D}dk = \frac{1}{2\sqrt{sD}}e^{-|x|\sqrt{s/D}}\end{aligned}$$

By the formula (4) , we obtain

$$\tilde{f}(x,s) = e^{-|x|\sqrt{s/D}} \tag{6}$$

**Note:** The Cauchy Distribution

$$p(x) = \frac{a}{\pi}\frac{1}{a^2+x^2}$$

has the Fourier transform

$$\hat{p}(k) = e^{-a|k|}$$

**Note:** Just like the Fourier transform, the Laplace transform of a PDF contains all the information of the moments of the PDF. This can be seen as follows:

$$\tilde{f}(x,s) = \int_0^{\infty} e^{-st}f(x,t)dt$$

Differentiating $n$ times with respect to $s$, and evaluating at $s=0$, we obtain

$$(-1)^n\frac{\partial^n \tilde{f}}{\partial s^n}(x,0) = \int_0^{\infty} t^n f(x,t)dt = \langle T(x)^n\rangle$$

in other words, the moments are the Taylor coefficients of the Laplace transform at $s=0$, multiplied by $(-1)^n$.

In our example, however, the Laplace transform $\tilde{f}(x,s)$ is not analytic, since $\frac{\partial \tilde{f}}{\partial s}(x,0)$ does not exist. This implies that $\langle T(x)\rangle = \infty$. On the other hand we have

$$\tilde{f}(x,0) = \int_0^{\infty} f(x,t)dt = 1 = \text{"eventual hitting probability"}$$

So the diffusion process will eventually go from 0 to $x$ with probability 1, but the expected time that this takes is infinity.

This might at first sight be surprising, but it should not be. Because there are so many trajectories with non-trivial likelihood which spend long long times in the wrong half axis (the half axis not containing $x$), an average over the hitting time might very well turn out to be infinite, and (6) shows that it does.

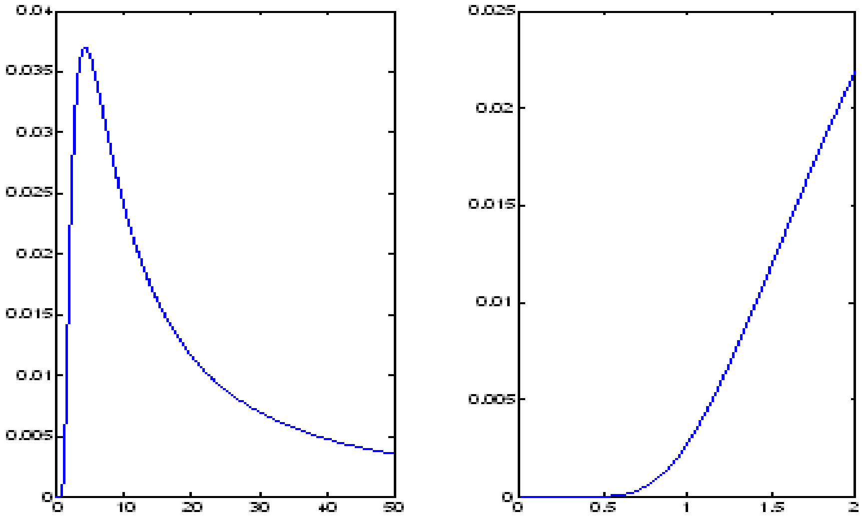


Figure 4: Seen above are two plots of the Smirnov function for values $x = 5$, $D = 1$. The ”fat tail”, or the powerlaw amplitude is apparent on the first plot, whereas the second plot focuses on the essential singularity at $t = 0$.

One might be interested in knowing more than the moments, and try to invert the Laplace transform given by (6) . This is possible and the resulting PDF is known as the Smirnov density:

$$f(x,t) = \frac{x}{\sqrt{4\pi D t^3}} \exp\left(-\frac{x^2}{4Dt}\right) \tag{7}$$

Note that $f(x,t)$ has an essential singularity at $t = 0$, indeed, $f(x, t \to 0)$ approaches 0 with its derivatives of all orders approaching zero. Since $f(x,t)$ is not identically zero, this is an indication of an essential singularity at $t = 0$. This is also obvious from the non-terminating Laurent series of $f(x,t)$.

The function $f(x,t)$ takes on its maximum at $t = \frac{1}{6}\frac{x^2}{D}$, which deserves the name "typical time" for reaching $x$ for the first time. However, since $f(x,t)$ has a power-law tail scaling like $At^{-3/2}$, we see that

$$\langle T(x)\rangle \sim \int_0^\infty tAt^{-3/2}dt \sim 2At^{1/2}|^\infty = \infty$$

verifying our earlier analysis.

The Smirnov density (7) is the only "one-sided Levy-stable distribution"[2] which can be expressed in terms of elementary functions.

[2] that is, it retains its shape upon addition of IID random variables with the same PDF as itself.

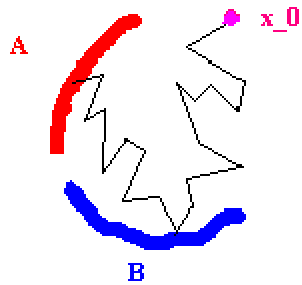


Figure 5: Illustration of a random walker first reflected by a reflecting boundary, and then absorbed by an absorbing boundary

One may wonder whether or not it is possible to obtain a finite mean first-passage time for this problem by adding more random walkers exploring the whole axis to reach $x$. To be more precise, if $X_i$, $1 \le i \le N$ are $N$ random walkers, then it is natural to redefine $T(x)$ as being the shortest time in which at least one of the random walkers from the collection $X_i$ hit $x$. Equivalently, $T(x) = \min T_i(x)$, wherer $T_i(x)$ is the first-passage time for the random walker $X_i$.

It is curious fact that one obtains a finite $T(x)$ for $N \ge 3$. Whenever finite, $T(x)$ is calculated to be proportional to the familiar typical time $x^2/D$.

# 4 General Formulation

In this section, we give a quick introduction to the general case of first-passage problems where the walkers blissfully live in spaces with dimension $d \ge 1$.

An absorbing boundary $A$ is a region of the domain where we do or do not want the random walker to hit. It is like the edge of the roof in the sleep walker example-once the random walker hits there, the game is over-the random walker stops walking. To obtain the PDF for the walker at $x$ at time $t$ having not hit the absorbing boundary $A$ for any time $t' < t$, we need to solve the Fokker-Planck equation with an "absorbing" boundary condition:

$$P(\mathbf{x}, t) = 0 \text{ for all } \mathbf{x} \in A, \text{and all } t$$

In the generic case, there will be nonzero probability current, which is $\mathbf{J} \cdot \hat{\mathbf{n}}$, flowing into the absorbing boundary $A$. In this case, we will have

$$\Pr\{\text{Having not hit } A\} = \int P(\mathbf{x}, t) d\mathbf{x} < 1$$

where the integration is taken over the whole domain.

One can also model a "reflecting" or "bouncing" boundary condition on a boundary $B$ by imposing

$$\mathbf{J}\cdot\hat{\mathbf{n}} = 0, \text{ for all } \mathbf{x}\in B$$

where $\hat{\mathbf{n}}$ is the unit normal to $B$, which comes to say that there is no probability flux through $B$. This corresponds to a random walker turning backwards, or being reflected, upon hitting the set $B$.

Let's summarize what we have said so far. In presence of both absorbing and reflecting boundaries, and a random walker starting from a point $\mathbf{x}_0$ at time $t=0$, all we need to do is to solve the Fokker Planck Equation, which is

$$\frac{\partial P}{\partial t} + \nabla\cdot\mathbf{J} = 0$$

where

$$J_i(\mathbf{x},t) = D_1^i(\mathbf{x},t)P(\mathbf{x},t) - \frac{\partial}{\partial x_i}\left(D_2^{ij}(\mathbf{x},t)P(\mathbf{x},t)\right)$$

is the probability current and $D_1$, $D_2$ are the coefficients in the FP equation as introduced before. This partial differential equation is supplied with the initial condition

$$P(x,0) = \delta(x - x_0)$$

which encourages the notation

$$P(\mathbf{x},t|\mathbf{x}_0,0) = P(x,t)$$

The boundary conditions are given by

$$P(x,t|x_0,0) = 0, \text{for all } x\in A,\ t\in\mathbb{R}^+$$

$$\mathbf{J}(\mathbf{x},t)\cdot\hat{\mathbf{n}} = 0, \text{ for all } \mathbf{x}\in B$$

If we denote the probability of "surviving", or not hitting $A$, up till time $t$ by $\mathcal{S}(\mathbf{x}_0,t)$, then we can write down

$$\mathcal{S}(\mathbf{x}_0,t) = \int P(\mathbf{x},t|\mathbf{x}_0,0)d\mathbf{x}$$

where the integration is taken over the whole space. The relation between this quantity and the rather familiar first-passage time $T(\mathbf{x}_0)$, which now denotes the time of first passage from $\mathbf{x}_0$ to $A$, as opposed from 0 to $\mathbf{x}_0$, is

$$\mathcal{S}(\mathbf{x}_0,t) = \Pr\{T(\mathbf{x}_0) > t\} = \int_t^\infty f(\mathbf{x}_0,t)dt$$

where $f(\mathbf{x}_0,t)$ is the PDF of $T(\mathbf{x}_0)$. Differentiating, we obtain

$$f(\mathbf{x}_0,t) = -\frac{\partial}{\partial t}\mathcal{S}(\mathbf{x}_0,t). \tag{8}$$

This is a good place to stop, next time Prof. Bazant will be out of town, but instead Chris will be talking about the discrete first passage processes. See you next Tuesday. We refer the curious reader to the references given at the end for a deeper understanding of the subject.

## References

[1] S.Redner, A Guide to First Passage Processes (Cambridge, 2001)

[2] H.Risken, The Fokker-Planck Equation (Springer, 2nd ed., 1989)

[3] K.Titievski, Lecture 14:Applications in Statistical Mechanics, 2005.