Asymptotics Outside the Central Region

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Until now we have focused on "typical" outcomes for a random walk in the "central region" of the probability distribution, which contains all of its weight as the number of steps N tends to ∞. For any finite N, now matter how large, however, there is always some chance of finding the random walk outside the central region. Such "extreme events" are not controlled by the Central Limit Theorem, and intead their probabilities depend sensitively on the "tails" of the step distribution.

There is a special class of distributions for which the probabilities of extreme events follow universal law – those with "fat" power-law tails, discussed in the previous lecture. In the first part of this lecture, we explain intuitively why, in a random walk with fat-tailed IID steps, the power-law tail is preserved in the final distribution. We also sketch a proof of the additivity of the power-law tail amplitude, analogous to a (divergent) cumulant, by investigating the singularity in the characteristic function.

In the second part, we move on to give a brief exposition of Laplace's method and its generalization to the complex plane (the steepest-descent or saddle-point method). We show how the latter can be used to obtain uniformly valid asymptotic approximations for the PDF of a random walk, with an analytic characteristic function. The approximation is valid even far outside the central region, provided that N is sufficiently large (although, as we will see in the next lecture, it is not always a good approximation for small N).

1 Additivity of power-law tail amplitudes

1.1 Intuitive explanation

Consider a probability distribution function (PDF) $p(x)$ with

$$p(|x|) \sim \frac{A}{|x|^{1+\alpha}} \text{ as } |x| \to \infty \tag{1}$$

but still assume $\langle x^2 \rangle < \infty$ so that the central limit theorem (CLT) applies. In the previous lecture, we showed that the "tail amplitude" is additive. In other words, the PDF the random walk with independent identical steps with PDF p has the PDF

$$P_N(|x|) \sim N \frac{A}{|x|^{1+\alpha}} \text{ as } |x| \to \infty$$

This is also true for the cumulative probability distribution (CDF), i.e.

$$\Pr(X_N > y) = \int_y^\infty P_N(x')dx' \underset{y\to\infty}{\sim} N \int_y^\infty p(x')dx' = N \Pr(x > y)$$

or, in case we have non-identical but still independent steps,

$$\Pr(X_N > y) \underset{y\to\infty}{\sim} \sum_{n=1}^{N} \Pr(x_n > y)$$

This fact can be interpreted intuitively as follows: When there is a "fat tail" in the displacement PDF $p(x)$, then the tail of the $P_N(x)$ for the sum X_N, is dominated by a single extreme event. The probability for such an event to occur in the course of N steps is approximately given by adding the (nearly independent) probabilities that each individual step is large enough to reach the extreme value. Of course, this argument is only asymptotically valid, for sufficiently large x.

1.2 A "high order" Tauberian theorem

In the previous lecture, we showed that the additivity of power-law tails follows if the leading singular term in the cumulant-generating function, $\psi(k) = \log \hat{p}(k)$, as $k \to 0$, has a coefficient proportional to the tail amplitude of $p(x)$, as $x \to \infty$. We made a conjecture along the lines of the following "theorem" and showed that it is satisfied by several examples. We leave a rigorous proof (or disproof and revision) to the reader, and below we sketch a partial derivation.

Conjecture. Consider a symmetric PDF $p(x)$ with a finite variance, σ^2, and let $\psi(k) = \log \hat{p}(k)$ be its cumulant genarating function. Then, for $\alpha > 2$ (not an even integer[1]):

$$p(x) \sim \frac{A}{x^{1+\alpha}}, \quad \text{as } x \to \infty \tag{2}$$

if and only if

$$\psi(k) = \log \hat{p}(k) \sim f(k) + g(k), \quad \text{as } k \to 0, \tag{3}$$

where

$$\begin{aligned} f(k) &\sim -\frac{\sigma^2 k^2}{2}, \quad \text{is analytic,} && (4)\\ g(k) &\sim c|k|^\alpha \quad \text{is singular, and} && (5)\\ c &= \frac{A\pi}{\Gamma(\alpha+1)\sin(\pi\alpha/2)}. && (6) \end{aligned}$$

Results of this type, which relate the large x power-law decay of a function to the small k singular power-law behavior of its Fourier (or Laplace) or Laplace transform, are called "Tauberian theorems". For a thorough introduction to Tauberian theorems for the Laplace transform in the context of continuous-time random walks (also discussed later in this class), see the excellent book of George Weiss [6]. Some Tauberian theorems are also discussed by Barry Hughes [3], in the context of Lévy distributions, which are also in the class notes [4]. These results, however, focus on the case of an infinite variance, $0 < \alpha < 2$, where $\psi(k) = \log \hat{p}(k)$ is singular at leading order, but here we claim a similar "high order" result for the case of a finite variance ($\alpha > 2$), where $\psi(k) \sim f(k)$ is analytic at leading order. Such results may also appear in the literature, but are beyond the scope of this class (and the knowledge of the professor).

1.3 Sketch of the Proof

We consider only the reverse implication (3) $\Rightarrow$ (2) and begin with the definition:

$$p(x) = \int_{-\infty}^{\infty} e^{ikx} e^{\psi(k)} \frac{dk}{2\pi} \tag{7}$$

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[1] If $\alpha > 2$ is a positive even integer, then the leading order behavior of $g(k)$ in Eq. (5) may contain extra logarithmic factors.

For $\hat{p}(k) = e^{\psi(k)}$ takes its maximum at $k = 0$, and typically drops quickly as k gets away from the origin, and the integrand of (7) does not have any *stationary points* (see [1] or [2] for details) the main contribution to the integral comes from a neighborhood of $k = 0$. We therefore expand ψ around $k = 0$,

$$\psi(k) = -\frac{k^2}{2} + c|k|^\alpha + h(k)$$

where $h(k)$ contains the higher order terms in k, then the integral takes to form

$$\begin{aligned} p(x) &\sim \int_{-\epsilon}^{\epsilon} e^{ikx} e^{-k^2/2} e^{c|k|^\alpha} e^{h(k)} \frac{dk}{2\pi} \\ &\sim \int_{-\infty}^{\infty} e^{ikx} e^{-k^2/2} (1 + c|k|^\alpha + ...)(1 + h(k) + ...) \frac{dk}{2\pi} \end{aligned}$$

We now remember our earlier calculations, namely, for integer m, we have

$$\int e^{ikx - k^2/2} k^m \frac{dk}{2\pi} = \frac{1}{\sqrt{2\pi}} H_m(x) e^{-x^2/2} \underset{x\to\infty}{\sim} \frac{1}{\sqrt{2\pi}} x^m e^{-x^2/2} = O(x^m e^{-x^2/2})$$

and it can also be shown that for any number $\alpha > -1$,

$$\begin{aligned} \int_{-\infty}^{\infty} e^{ikx - k^2/2} |k|^\alpha \frac{dk}{2\pi} &= \frac{1}{\pi} \int_0^\infty \cos(kx) e^{-k^2/2} |k|^\alpha \frac{dk}{2\pi} \\ &\underset{x\to\infty}{\sim} \frac{\Gamma(\alpha+1)\sin(\pi(\alpha+1)/2)}{\pi x^{1+\alpha}} \end{aligned}$$

We can verify this for the odd integer $\alpha = m \geq 3$, by using the results from the previous lecture

$$\int_0^\infty \cos(kx) k^m e^{-k^2/2} dk = -\frac{d^m}{dx^m}[\sqrt{2} D(\frac{x}{\sqrt{2}})]$$

Keeping in mind that the Dawson integral has

$$D(z) \underset{z\to\infty}{\sim} \frac{1}{2z}$$

we obtain

$$\begin{aligned} \int_0^\infty \cos(kx) k^m e^{-k^2/2} dk \underset{x\to\infty}{\sim} -\frac{d^m}{dx^m}(\frac{1}{x}) &= \frac{(-1)^{m+1} m!}{x^{m+1}} \\ &= \frac{\sin(\pi m/2)\Gamma(m+1)}{x^{m+1}} \end{aligned}$$

2 Uniform Approximations from Saddle Point Asymptotics

For a random walk with IID steps with $\psi(k)$, we have

$$P_N(x) = \int_{-\infty}^{\infty} e^{ikx} e^{N\psi(k)} \frac{dk}{2\pi} \tag{8}$$

In case $\psi(k)$ is an analytic function, this integral may be evaluated accurately by making use of the saddle point method. In the coming subsections, we give a brief introduction to the saddle point method and show how to use it to obtain accurate approximations to the integral (8). For a detailed treatment of the saddle point method, we refer the reader to [1] and [2]. For nice pictures, see also [5].

Note that it is not always possible to make use of saddle point method, because $\psi(k)$ may not be analytic, say, due to a fat tail in the PDF $p(x)$, as we have demonstrated in the previous section.

2.1 The Laplace's method

Here we first give a brief exposition of the Laplace's method. The Laplace's method is used to find the asymptotics of the integrals of the form

$$\int e^{-N\psi(k)}dk \tag{9}$$

as $N \to \infty$. The idea is that as N gets larger and larger, the dominant contribution to the integral (9) comes typically from a single point, which is the point k_0 at which $\psi(k)$ attains its minimum. This is because as $N \to \infty$, the value of the integrand outside k_0 becomes exponentially small compared to the value of the integrand at k_0.

That being said, the value of the integral (9) depends on the behaviour of the function $\psi(k)$ in a small neighborhood of the point k_0. We can use the Taylor Series of $\psi(k)$ expanded around k_0 in the calculation of the integral (9) without incurring much error. To summarize, the Laplace's method has the following steps.

1. Locate the minimum of the function $\psi(k)$. Argue that the dominant contribution of the integral (9) comes from a neighborhood of this point in the large N limit. In other words, write down

$$\int_a^b e^{-N\psi(k)}dk \approx \int_{k_0-\epsilon}^{k_0+\epsilon} e^{-N\psi(k)}dk \tag{10}$$

for a small number $\epsilon > 0$. Here, we are assuming that k_0 is an interior point of $[a, b]$, but the case k_0 is an end point can be handled similarly.

2. Argue that, for $\epsilon > 0$ small, the function $\psi(k)$ can be approximated by its Taylor expansion at k_0, in symbols

$$\psi(k) = \psi(k_0) + (k - k_0)\psi'(k_0) + \frac{1}{2}(k - k_0)^2\psi''(k_0) + O((k - k_0)^3)$$

Note that in case k_0 is an interior point, we have $\psi'(k_0) = 0$, and typically $\psi''(k_0) \neq 0$. Let's assume for simplicity that this is the case. Plug this series into the integral on the right hand side of (10). Obtain

$$\int_a^b e^{-N\psi(k)}dk \approx e^{-N\psi(k_0)} \int_{k_0-\epsilon}^{k_0+\epsilon} e^{-\frac{1}{2}N(k-k_0)^2\psi''(k_0)}dk \tag{11}$$

3. Argue that, since the integrals

$$\int_{k_0+\epsilon}^{\infty} e^{-\frac{1}{2}N(k-k_0)^2\psi''(k_0)}dk, \quad \int_{-\infty}^{k_0-\epsilon} e^{-\frac{1}{2}N(k-k_0)^2\psi''(k_0)}dk$$

are exponentially small compared to the integral on the right hand side of (11), we can write

$$\begin{aligned} \int_a^b e^{-N\psi(k)}dk &\approx e^{-N\psi(k_0)} \int_{k_0-\epsilon}^{k_0+\epsilon} e^{-\frac{1}{2}N(k-k_0)^2\psi''(k_0)}dk \\ &\approx e^{-N\psi(k_0)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}N(k-k_0)^2\psi''(k_0)}dk \end{aligned} \tag{12}$$

4. Now the rightmost integral in (12) can be found using the fact that

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

All in all, we obtain

$$\int_a^b e^{-N\psi(k)}dk \approx e^{-N\psi(k_0)}\sqrt{\frac{2\pi}{N\psi''(k_0)}} \text{ as } N \to \infty$$

In case $\psi''(k_0) = 0$, the steps $1-3$ can still be repeated, but at step 4 one will need to know the value of one of the integrals

$$\int_{-\infty}^{\infty} e^{-x^n} dx$$

which can be evaluated in terms of gamma functions. For reference,

$$\int_{-\infty}^{\infty} e^{-x^3} dx = \frac{\Gamma(1/3)}{3}$$

2.2 The Saddle Point Method

The saddle point method, which is basically an extension of the Laplace method to the integrals over the contours in the complex plane, deals with the integrals of the type

$$\int_C e^{-N\psi(z)} dz = \int_C e^{-N\,\mathrm{Re}\,\psi(z)} e^{-iN\,\mathrm{Im}\,\psi(z)} dz \tag{13}$$

where now $\psi(z)$ is a complex valued analytic function of the complex variable z. By the same token as in the previous section, the dominant contribution to the integral in the $N \to \infty$ limit comes from the point z_1 where $|e^{-\psi(z)}| = e^{-\mathrm{Re}\,\psi(z)}$ attains its maximum on the path C. While this is true, there may still be large cancellations within the part of the path C containing z_1, due to the factor $e^{-iN\,\mathrm{Im}\,\psi(z)}$ in the integrand. To overcome this issue, we will invoke the Cauchy's theorem and deform the path C of integration to a new one C' such that

(A) Along C', $\mathrm{Re}\,\psi(z)$ attains its minimum at a point z_0, and grows quickly as we move away from z_0 along C'

(B) In at least a small section of C' containing z_0, $\mathrm{Im}\,\psi(z)$ stays constant so that there are no non-trivial cancellations

Seen in the light of the previous section on Laplace's method, the integral along C' has its dominant contribution from a neighborhood of the point z_0 along C', and can be evaluated by repeating the steps 1-4 of the previous section.

On the other hand, a point z_0 which satisfies the conditions given in (A) and (B) must satisfy $\psi'(z_0) = 0$. The reason is follows: Because of (A), the derivative of $\mathrm{Re}\,\psi(z)$ at z_0 taken along the curve C' is zero. Because of (B), the derivative of $\mathrm{Im}\,\psi(z)$ at z_0 taken along the curve C' is zero. Combining these, we see that the derivative of $\psi(z)$ at z_0 taken along the curve C' is zero. But since $\psi(z)$ is analytic, this means $\psi'(z_0) = 0$.

Conversely, it is true that if $\psi'(z_0) = 0$, then a path C' which passes through the point z_0 and satisfying the properties in (A) and (B) can be found. In this case, C' is called the path of steepest descent for the integrand in (13).

A point z_0 for which $\psi'(z_0) = 0$ is called a saddle point. Because ψ is analytic, the real valued functions $|\psi|$, $\mathrm{Re}\,\psi$, $\mathrm{Im}\,\psi$ cannot have any extrema in the interior of a region in the complex plane. Therefore $\psi'(z_0) = 0$ implies that the graph of the those functions a saddle-type near the point z_0, justifying the terminology.

We summarize the procedure for the saddle point for the integral in (13) method as follows:

1. Locate saddle points by solving $\psi'(z_0) = 0$.

2. Investigate how the original path C can be deformed so that it passes through some of the saddle points along their steepest descent paths (i.e. paths over which $\mathrm{Im}\,\psi(z) =$ constant) at least locally. In other words, find the saddle points which "contribute" to the integral.

3. Go through the steps given in the previous section on Laplace's method. Assuming that there is a single saddle point z_0 which contributes, and $\psi''(z_0) = 0$, these steps are briefly

$$\begin{aligned}\int_C e^{-N\psi(z)}dz &\sim e^{-N\psi(z_0)}\int_{C'} e^{-N\frac{1}{2}\psi''(z_0)(z-z_0)^2}dz \\ &\sim e^{-N\psi(z_0)}\int_{-\infty}^{\infty} e^{-N\frac{1}{2}|\psi''(z_0)|t^2}e^{-\frac{1}{2}i\arg\psi''(z_0)}dt \\ &= e^{-N\psi(z_0)}e^{-\frac{1}{2}i\arg\psi''(z_0)}\sqrt{\frac{2\pi}{N|\psi''(z_0)|}}\end{aligned}$$

by making the change of variables $t = e^{\frac{1}{2}i\arg\psi''(z_0)}(z - z_0)$ in the second step. Indeed, when t is real,

$$\begin{aligned}\mathrm{Im}(\psi(z)) &\sim \mathrm{Im}(\psi(z_0) + \frac{1}{2}\psi''(z_0)(z-z_0)^2) \\ &= \mathrm{Im}\,\psi(z_0) + \mathrm{Im}\{\frac{1}{2}\psi''(z_0)[e^{-\frac{1}{2}i\arg\psi''(z_0)}t]^2\} \\ &= \mathrm{Im}\,\psi(z_0) = \text{constant}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Re}(\psi(z)) &\sim \mathrm{Re}(\psi(z_0) + \frac{1}{2}\psi''(z_0)(z-z_0)^2) \\ &= \mathrm{Re}\,\psi(z_0) + \mathrm{Re}\{\frac{1}{2}\psi''(z_0)[e^{-\frac{1}{2}i\arg\psi''(z_0)}t]^2\} \\ &= \mathrm{Re}\,\psi(z_0) + \frac{1}{2}|\psi''(z_0)|t^2\end{aligned}$$

hence t being real corresponds to the direction along which ψ experiences the steepest descent.

2.3 Application to the Random Walks

We will obtain asymptotic approximations that are valid outside the central region (more precisely, valid when $x = O(N)$ and as $N \to \infty$) for random walks by making use of the saddle point method. To this end, we start with

$$P_N(x) = \int_{-\infty}^{\infty} e^{ikx}e^{N\psi(k)}\frac{dk}{2\pi} = \int e^{-Nf(k,\xi)}\frac{dk}{2\pi}$$

where

$$\begin{aligned}f(k,\xi) &= -ik\xi + \psi(k) \\ \xi &= \frac{x}{N}\end{aligned}$$

We then make use of the saddle point method and obtain an approximation for $N \to \infty$, $\xi = \frac{x}{N} = O(1)$ considered fixed. Therefore this approximation is valid for $x = O(N)$. The central region, which is usually confined to $x = O(\sqrt{N})$, can be recovered by considering $\xi = O(\frac{1}{\sqrt{N}})$.

References

[1] Bender, C. and Orszag, S., Advanced Mathematical Methods for Scientist and Engineers, Mc-Graw-Hill, 1978

[2] Cheng, H., Advanced Analytic Methods in Continuum Mathematics, Fundementals for Science and Engineering, Luban Press 2004.

[3] Hughes, B., Random Walks and Random Environments, Vol. 1 (Oxford, 1996).

[4] Randall, G., Really Fat Tails (Levy Flights), PS notes, lecture 12, 2003, http://www-math.mit.edu/18.366/lec03/lecture12.pdf

[5] Savva, N., Saddle Point Asymptotics Outside the Central Region, PS notes, lecture 7, 2003, http://www-math.mit.edu/18.366/lec03/lecture07.pdf

[6] G. H. Weiss, Aspects and Applications of the Random Walk (Elsevier, 1994).