

Lecture 4: Asymptotics in the Central Region

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Lecture 3 provided a derivation of the Central Limit Theorem - showing that a distribution $P_N(\mathbf{x})$ tends to the multivariate normal distribution in the central region for sufficiently large N . This derivation was predicated on ignoring certain higher order terms of the expansion used in Laplace's method. Lecture 4, then, examines the form of these higher order terms and under what conditions these terms become non-negligible.

1 Gram-Charlier Expansions

Much of the Gram-Charlier derivation was covered in the previous lecture. To see a more complete description of the derivation, see the notes for lecture 3. The scaled variable, Z_N , used in the following is given by:

$$Z_N = \frac{X_N - N\mu}{\sigma\sqrt{N}} \quad (1)$$

For each iid step, we make two requirements on the moments of the distribution.

$$c_2 = \sigma^2 < \infty \quad (2)$$

and, for an expansion up to order m ,

$$\langle |X|^m \rangle < \infty \quad (3)$$

These requirements of non-divergence allow an expansion for the distribution $\phi_N(x)$, the PDF of Z_N as $N \rightarrow \infty, Z \sim O(1)$. See lecture 3 for detailed derivation.

$$\phi_N(Z) \sim \phi(Z) \left[1 + \frac{h_3(Z)}{\sqrt{N}} + \frac{h_4(Z)}{N} + \dots + \frac{h_m(Z)}{N^{\frac{m}{2}-1}} \right] \quad (4)$$

where

$$\phi(Z) = \frac{e^{-\frac{Z^2}{2}}}{\sqrt{2\pi}} \quad (5)$$

$$h_3(Z) = \frac{\lambda_3}{3!} H_3(Z) \quad (6)$$

$$h_4(Z) = \frac{\lambda_4}{4!} H_4(Z) - \frac{1}{2} \left(\frac{\lambda_3}{3!} \right)^2 H_6(Z) \quad (7)$$

$H_n(Z)$'s are the Hermite polynomials, defined by the relationship

$$(-1)^n H_n(Z)\phi(Z) = \frac{d^n}{dZ^n}\phi(Z) \quad (8)$$

λ_n are called the “normalized cumulants” and are defined by

$$\lambda_n = \frac{c_n}{\sigma^n} \quad (9)$$

The fact that $Z_N \sim O(1)$ implies that $(X_N - N\mu) \sim O(\sigma\sqrt{N})$. The region in which this approximation is expected to hold is called the “central region”. For rare events (i.e. events located in the tails of the distribution), the CLT and the associated Gram-Charlier expansion break down. (For an interesting discussion of rare events and the limits of the CLT in the context of finance, see J.P. Bouchaud, *Theory of Financial Risks* (2000) in the recommended reading.)

A similar expansion can be created for the cumulative distribution function, $\Phi_N(Z)$

$$\Phi_N(Z) = \int_{-\infty}^Z \phi_N(Z')dZ' \equiv \Pr(Z_N < Z) \quad (10)$$

$$\Phi_N(Z) \sim \Phi(Z) + \phi(Z) \left[\frac{G_3(Z)}{\sqrt{N}} + \frac{G_4(Z)}{N} + \dots \right] \quad (11)$$

where

$$\Phi(Z) = \int_{-\infty}^Z \phi(Z')dZ' \quad (12)$$

$$G_3(Z) = -\frac{\lambda_3}{3!}H_2(Z) \quad (13)$$

$$G_4(Z) = -\frac{\lambda_4}{4!}H_3(Z) + \frac{\lambda_3^2}{2(3!)^2}H_5(Z) \quad (14)$$

Note: For more information, see W. Feller, *Introduction to Probability Theory* Vol. II (1971) who refers to this as an “Edgeworth Expansion” (in one dimension). For the more general context of random walks in d dimensions, see B. Hughes, *Random Walks and Random Environments*, Vol. I, Ch. 2 (Oxford, 1996). Hughes attributes to Chebyshev (1860) the first to study convergence to the CLT and uses the term “Gram-Charlier expansion” in place of “Edgeworth expansion” for random walks.

The following is a rigorous result related to the Gram-Charlier expansions above, which remarkably holds for *all* $N \geq 1$ and z , although the bound is most accurate in the central region for $N \gg 1$.

Theorem. (Berry-Esséen) If $m_1 = 0$, $m_2 = \sigma^2$, and $\bar{m}_3 = \langle |x|^3 \rangle < \infty$, then

$$|\Phi_N(z) - \Phi(z)| < \frac{3\bar{m}_3}{\sigma^3\sqrt{N}} \quad (15)$$

is a uniform bound for all $N \geq 1$ and z .

To conclude this section, we give Gram-Charlier expansions for isotropic random walks of fixed step size a in two and three dimensions. The appropriate scaled coordinate here is:

$$\vec{Z}_N = \frac{\vec{X}_N}{a\sqrt{N}} \quad (16)$$

The two-dimensional case was first derived by Pearson in 1906:

$$\phi_N(\vec{Z}) \sim \frac{1}{\pi} e^{-Z^2} \left[1 - \frac{1}{4N} (2 - 4Z^2 + Z^4) + \frac{1}{12N^2} \left(1 - 12Z^2 + 15Z^4 - 14Z^6 + \frac{3}{8}Z^8 \right) + \dots \right] \quad (17)$$

Notice that there is no \sqrt{N} term in the expansion. This is because there is no drift due to unequal step probabilities, and the x^3 term drops out. The three-dimensional case was first derived by Rayleigh in 1919:

$$\phi_N(\vec{Z}) \sim \frac{3^{\frac{3}{2}}}{2\pi} e^{-\frac{3Z^2}{2}} \left[1 - \frac{3}{20N} (5 - 10Z^2 + 3Z^4) + \frac{1}{40N^2} \left(\frac{29}{4} - 69Z^2 + \frac{981}{10}Z^4 - \frac{1341}{35}Z^6 + \frac{81}{20}Z^8 \right) + \dots \right] \quad (18)$$

2 Width of the Central Region

Now, we estimate where the CLT approximation breaks down by considering one dimension ($d=1$)

$$\phi_N(Z) \sim \phi(Z) \left[1 + \frac{h_3(Z)}{\sqrt{N}} + \frac{h_4(Z)}{N} + \dots \right] \quad (19)$$

Case (i): $c_3 < \infty$, $\lambda_3 \neq 0$. The CLT begins to break down when $\frac{h_3(Z)}{\sqrt{N}} \sim O(1)$. To determine the Z value at which the CLT breaks down, we find Z_{max} such that $\frac{h_3(Z_{max})}{\sqrt{N}} \simeq 1$. We also note at this time that the Hermite polynomial $H_n(Z)$ can be approximated as $H_n(Z) \sim Z^n$ as $Z \rightarrow \infty$.

$$\frac{h_3(Z_{max})}{\sqrt{N}} \simeq \frac{\lambda_3 Z_{max}^3}{3!\sqrt{N}} \quad (20)$$

$$\Rightarrow Z_{max} \simeq \left(\frac{6}{\lambda_3} \right)^{\frac{1}{3}} N^{\frac{1}{6}} \quad (21)$$

or, switching back to X representaion

$$\frac{|X_N - N\mu|_{max}}{\sigma} \simeq \left(\frac{6}{\lambda_3} \right)^{\frac{1}{3}} N^{\frac{2}{3}} \quad (22)$$

Although we expect this diffusion process to scale as $N^{\frac{1}{2}}$, the width of the CLT approximation region scales as $N^{\frac{2}{3}}$. Thus, we anticipate that, if λ_3 is finite, the CLT approximation will be quite good.

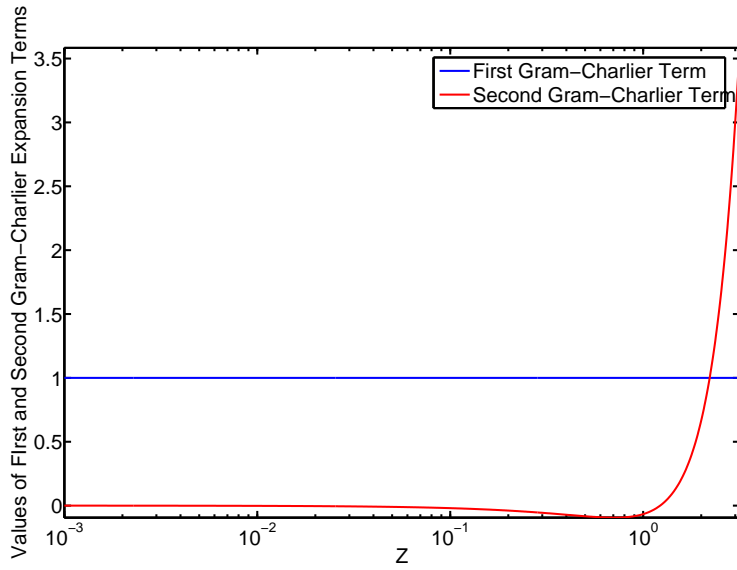


Figure 1: Comparison of relative magnitudes of the first and second terms of the Gram-Charlier expansion for $N = 100$ and $\lambda_3 = 1$

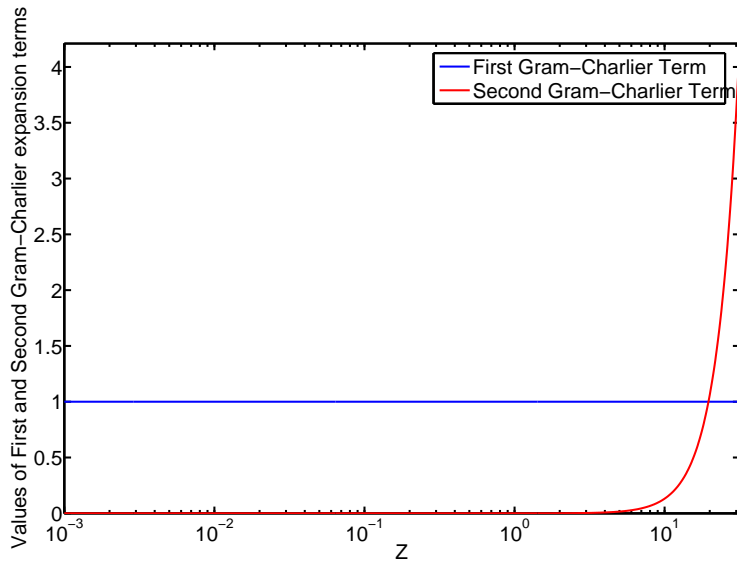


Figure 2: Comparison of relative magnitudes of the first and second terms of the Gram-Charlier expansion for $N = 1e^8$ and $\lambda_3 = 1$

This Z_{max} relationship can be confirmed visually, as shown in Figures 1 and 2. These figures compare the values of the first term of the Gram-Charlier expansion (constant value 1) with the second term $\frac{h_3(Z)}{\sqrt{N}}$ for the cases of $N = 100$ and $N = 1e^8$ respectively. For the case of $\lambda_3 = 1$, the predicted values of Z_{max} are $O(1)$ and $O(10)$ respectively. Figures 1 and 2 confirm that in these regions, the second term of the Gram-Charlier expansion become important, and can no longer be ignored;

Case (ii): If $\lambda_3 = 0$ (e.g. for an isotropic random walk), but $\lambda_4 < \infty$, then the relevant range of approximation for the CLT is defined by the relation $\frac{h_4(Z_{max})}{N} \sim 1$. Following the same procedure as in Case (i), we obtain the relation:

$$\frac{|X_N - N\mu|_{max}}{\sigma} \simeq \left(\frac{24}{\lambda_4}\right)^{\frac{1}{4}} N^{\frac{3}{4}} \tag{23}$$

The width of the central region in Case (ii) is $O(N^{\frac{3}{4}})$, even better than in Case (i).

3 Fat Tails

We say that a PDF has a “fat tail” if it has divergent moments, as in the simple case of a power-law tail of the form:

$$p(x) \sim \frac{A}{x^n} \text{ as } x \rightarrow \infty \tag{24}$$

where $m_l = \infty$ for $l \geq n - 1$. For cases in which m_2 exists ($n > 3$), the CLT will still hold. However, as some higher moments are divergent, the expansion approach of Section 1 will eventually fail to hold.

Consider the fat-tailed PDF

$$p(x) = \frac{\sqrt{2}}{\pi} \frac{1}{1+x^4} \tag{25}$$

In this example, $\langle |x|^\alpha \rangle = \infty$ for $\alpha \geq 3$. We expect the CLT to hold in the central region (since σ exists), but the correction terms will be different.

$$p(\hat{k}) = \int e^{-ikx} p(x) dx = \frac{\sqrt{2}}{\pi} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{1+x^4} dx \tag{26}$$

$$p(\hat{k}) = e^{-|k|/2} \left(\cos\left(\frac{k}{2}\right) + \sin\left(\frac{|k|}{2}\right) \right) \tag{27}$$

The Fourier transform of $p(x)$ is no longer analytic. As $k \rightarrow 0$,

$$p(\hat{k}) \sim 1 - \frac{k^2}{2} - \frac{|k|^3}{3\sqrt{2}} + \frac{k^4}{6} - \dots \tag{28}$$

Where the first two terms define the analytic portion of the expansion, and subsequent terms the nonanalytic. One can also define an approximation to $\psi(k)$ as

$$\psi(k) = \log p(\hat{k}) \sim \frac{k^2}{2} - \frac{|k|^3}{3\sqrt{2}} + \dots \tag{29}$$

In the next lecture, we will see how the Gram-Charlier expansion must be modified in cases such as this, due to the singularity in the characteristic function. It will turn out that the power-law tail amplitude, A , plays the role of a (divergent) cumulant.