18.311 — MIT (Spring 2013)

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Problem Set # 05. Due: Fri. April 12.

Turn it in before 3:00 PM, in the box provided in Room 2-285.

IMPORTANT: The Regular and the Special Problems must be **stapled in TWO SEPARATE packages,** each with your **FULL NAME** clearly spelled.

Contents

1 Regular Problems.

1.1 Statement: Initial value problem with Q quadratic # 01a.

Consider the traffic flow equation

$$\rho_t + q_x = 0, \tag{1.1}$$

for a flow $q = Q(\rho)$ that is a quadratic function of ρ. In this case $c = dQ/d\rho$ is a conserved quantity, and the problem (including shocks, if any) can be entirely formulated in terms of c, which satisfies

$$c_t + \left(\frac{1}{2}\,c^2\right)_x = 0. \tag{1.2}$$

1. Consider the initial value problem determined by (1.2) and [1]

$$c(x,\,0) = 0 \;\text{ for }\; x \le 0 \quad \text{and} \quad c(x,\,0) = 2\,\sqrt{x} \;\text{ for }\; x \ge 0. \tag{1.3}$$

Without actually solving the problem, **argue that the solution to this problem must have the form**

$$c = t\,f(x/t^2) \;\text{ for }\; t > 0, \quad \text{for some function } f. \tag{1.4}$$

[1] In a traffic problem, c must satisfy $c(\rho_j) \le c \le c(0)$. Ignore the fact that this does not apply for (1.2).

Hint. Let $c = c(x,\, t)$ be the solution to the problem. For any constant $a > 0$, define $\mathcal{C} = \mathcal{C}(x,\, t)$ by $\boldsymbol{\mathcal{C} = \frac{1}{a}\, c(a^2\, x,\; a\, t)}$*. What problem does $\mathcal{C}$ satisfy? Use now the fact that the solution to (1.2–1.3) is unique to show that (1.4) must apply, by selecting a appropriately at any fixed time $t > 0$.*

2. Use the method of characteristics to **solve the problem in (1.2–1.3).** Write the solution **explicitly** for all $\boldsymbol{t > 0}$, and **verify that it satisfies (1.4).** *Warning: the solution involves a square root. Be careful to select the correct sign, and to* **justify your choice.**

3. For the solution obtained in item **2**, **evaluate $\boldsymbol{c_x}$ at $\boldsymbol{x = 0}$ for $\boldsymbol{t > 0}$.** Note that this derivative is discontinuous there, so it has two values.

1.2 Statement: Initial value problem with Q quadratic # 02a.

Consider the traffic flow equation

$$\rho_t + q_x = 0, \tag{1.5}$$

for a flow $q = Q(\rho)$ that is a quadratic function of ρ. In this case $c = dQ/d\rho$ is a conserved quantity, and the problem (including shocks, if any) can be entirely formulated in terms of c, which satisfies

$$c_t + \left(\frac{1}{2}\, c^2\right)_x = 0. \tag{1.6}$$

1. Consider the initial value problem determined by (1.6) and [2]

$$c(x,\, 0) = 0 \;\text{ for }\; x \leq 0 \quad \text{and} \quad c(x,\, 0) = -2\sqrt{x} \;\text{ for }\; x \geq 0. \tag{1.7}$$

Without actually solving the problem, **argue that the solution to this problem must have the form**

$$c = t\, f(x/t^2) \;\text{ for }\; t > 0, \quad \text{for some function } f. \tag{1.8}$$

Hint. Let $c = c(x,\, t)$ be the solution to the problem. For any constant $a > 0$, define $\mathcal{C} = \mathcal{C}(x,\, t)$ by $\boldsymbol{\mathcal{C} = \frac{1}{a}\, c(a^2\, x,\; a\, t)}$*. What problem does $\mathcal{C}$ satisfy? Use now the fact that the solution to (1.6–1.7) is unique to show that (1.8) must apply, by selecting a appropriately at any fixed time $t > 0$.*

2. Use the method of characteristics to **solve the problem in (1.6–1.7).** Write the solution **explicitly** for all $\boldsymbol{t > 0}$, and **verify that it satisfies (1.8).** Explicitly **describe the region in space-time where the solution by characteristics yields multiple-values.**
Hint 1: This problem leads to crossings of characteristics. Hence a shock is needed to avoid multiple-values. In order to correctly visualize the region in space-time where the solution by characteristic

[2] In a traffic problem, c must satisfy $c(\rho_j) \leq c \leq c(0)$. Ignore the fact that this does not apply for (1.6).

yields multiple values, I strongly suggest that you plot (for a typical $t > 0$ time) the multiple valued solution $c = c(x, t)$ — see note #1. From this plot it should become evident (i) what the region in space-time where the solution by characteristics yields multiple-values is. (ii) What the correct shock equation is (square roots are involved, and you need to pick, and **justify** *the correct sign).*
Note #1. For any fixed time $t > 0$, the solution by characteristics gives the solution curve (x, c) parametrically, in terms of the characteristic parameter ξ. That is: $x = x(\xi, t)$ and $c = c(\xi)$.
Hint 2. In the process of solving this problem you will get an ode that the shock path $x = \sigma(t)$ must satisfy. Note that you do not need the general solution for this equation, just the one that gives you the shock path. You should be able to extract the form of this particular solution from (1.8), which will then reduce the ode to an easy to solve algebraic equation.

2 Special Problems.

2.1 Statement: Information loss and traffic flow shocks.

In this problem we will show that shocks in traffic flow are associated with information loss, and will derive a formula that quantifies this loss. For simplicity, consider periodic solutions to the traffic flow equation

$$\rho_t + q_x = 0, \quad \text{periodic of period } T > 0, \tag{2.9}$$

where the traffic flux is given by $q = Q(\rho)$. Assume that there is a single shock per period, at $x = \sigma(t)$. The shock velocity, s, is given by the Rankine-Hugoniot jump condition

$$\frac{d\sigma}{dt} = s = \frac{q_R - q_L}{\rho_R - \rho_L}, \tag{2.10}$$

where ρ_R (resp.: ρ_L) is the value of ρ immediately to the right (resp.: the left) of the shock discontinuity, $q_R = Q(\rho_R)$, and $q_L = Q(\rho_L)$. In addition, the shock satisfies the *entropy condition*

$$c_L > s > c_R, \tag{2.11}$$

where $c = \dfrac{dQ}{d\rho}$ is the characteristic speed, $c_R = c(\rho_R)$, and $c_L = c(\rho_L)$. Now, let

$$\mathcal{I} = \mathcal{I}(t) = \int_{\text{period}} \frac{1}{2}\,\rho^2(x,\,t)\,dx. \tag{2.12}$$

1. Show that:

$$\frac{d\mathcal{I}}{dt} = \frac{1}{2}\,(q_L - q_R)\,(\rho_L + \rho_R) + \int_{\rho_L}^{\rho_R} \rho\,c(\rho)\,d\rho. \tag{2.13}$$

Hint. For any short enough time period, a constant "a" exists such that $a < \sigma(t) < a + T$ during the time interval. Given this, write $\mathcal{I}$ in the form $\mathcal{I} = \int_a^\sigma \frac{1}{2}\rho^2\,dx + \int_\sigma^{a+T} \frac{1}{2}\rho^2\,dx$, before taking the time derivative. Then use the fact that, in each of the two intervals ρ satisfies

$$\rho_t = -c(\rho)\,\rho_x = -\frac{1}{\rho}\,f_x \quad \textit{where} \quad f = \int_0^\rho s\,c(s)\,ds. \tag{2.14}$$

In addition, use (2.10) to eliminate $\dot{\sigma}$ from the resulting formulas.

2. **Show that:**

$$\frac{d\mathcal{I}}{dt} < 0 \tag{2.15}$$

by interpreting the right hand side in (2.13) geometrically, as (the negative of) the area between two curves — such area known to be positive. *Hint. Use $\rho\,c = \frac{d}{d\rho}(\rho\,q) - q$ and (2.17).*

Remark 2.1 *Recall that, in traffic flow, $q = Q(\rho)$ is a concave function,*[3] *vanishing at both $\rho = 0$ and the jamming density $\rho_j > 0$, positive for $0 < \rho < \rho_j$, with a maximum (the road capacity q_m) at some value $0 < \rho_m < \rho_j$. Because of this,* **the conditions (2.10–2.11) are equivalent to:**[4]

The shock velocity is given by the slope of the secant line joining $(\rho_L,\, q_L)$ to $(\rho_R,\, q_R)$. Further $\rho_L < \rho_R$. (2.16)

Note that **For $\rho_L < \rho < \rho_R$, the curve $q = Q(\rho)$ is *above* the secant line,** *(2.17)*
because Q is concave.

Remark 2.2 *How does $\mathcal{I}$ in (2.12) related to "information"? Basically, we argue that a function contains the least amount of information when it is a constant, and that the more "oscillations" it has, the more information it carries. A (rough) measure of this is given by*

$$\mathcal{J} = \int_{\text{period}} \frac{1}{2}\,(\rho - \overline{\rho})^2\,dx, \tag{2.18}$$

where $\overline{\rho}$ is the average value of ρ (note that conservation guarantees that $\overline{\rho}$ is a constant in time). Alternatively, write ρ the Fourier series $\rho = \sum_n \rho_n(t)\,e^{i\,n\,2\,\pi\,x/T}$ for ρ. Then the amount of "oscillation" in ρ can be characterized by $\sum_{n\neq 0} \frac{1}{2}\,|\rho_n|^2$, which is the same as (2.18). It is easy to see that

$$\mathcal{J} = \mathcal{I} - T\,\overline{\rho}^2. \tag{2.19}$$

Hence $\dot{\mathcal{J}} = \dot{\mathcal{I}}$, so that (2.15) indicates that the shock is driving ρ towards a constant value.

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[3] Concave means that $\frac{d^2Q}{d\rho^2} < 0$. Note also that **only solutions satisfying $0 \leq \rho \leq \rho_j$ are acceptable.**

[4] Thus shocks move slower than cars, and the density a driver sees increases as the car goes through a shock.

The notion of "information" that we use here is not the same as that in "information theory" (which requires a stochastic process of some kind). Neither is it the same as the related notion of "entropy" of statistical mechanics (for this we would need a statistical theory[5] *for traffic flow). Nevertheless,* $S = -\mathcal{J}$ *plays the same role for (2.9) that entropy plays for gas dynamics.*

Finally, we point out that it can be shown that: for any convex function $h = h(\rho)$ *(i.e.:* $h'' > 0$*)*

$$\frac{d}{dt}\int_{\text{period}} h(\rho)\,dx < 0 \quad \textit{when shocks are present,}$$

though showing this is a bit more complicated than for the special case $h = \frac{1}{2}\,\rho^2$.

THE END.

[5] Such theories exist.