18.311 — MIT (Spring 2013)

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Problem Set # 04. Due: Fri. April 5.

Turn it in before 3:00 PM, in the box provided in Room 2-285.

IMPORTANT: The Regular and the Special Problems must be **stapled in TWO SEPARATE packages,** each with your **FULL NAME** clearly spelled.

Contents

1 Regular Problems.

1.1 Statement: Haberman problem 77.01.

If $u = u_{\mathsf{max}}\,(1 - \rho/\rho_{\mathsf{max}})$, then what is the velocity of a traffic shock separating densities ρ_0 and ρ_1? Simplify the expression as much as possible. Show that the shock velocity is the average of the density wave velocities associated with ρ_0 and ρ_1.

1.2 Statement: Haberman problem 77.06.

Consider Burger's equation, as derived in exercise 77.5:

$$\rho_t + u_{\mathsf{max}} \left(1 - \frac{2\,\rho}{\rho_{\mathsf{max}}}\right) \rho_x = \nu\,\rho_{xx}. \tag{1.1}$$

Suppose that a solution exists as a density wave, moving without change of shape at velocity V

$$\rho = f(x - V\,t). \tag{1.2}$$

(a) What O.D.E. is satisfied by f?

(b) Integrate this differential equation once. By graphical techniques show that a solution exists, such that $\rho \to \rho_2$ as $x \to \infty$ and $\rho \to \rho_1$ as $x \to -\infty$, only if $\rho_2 > \rho_1$. Roughly sketch this solution, and give a physical interpretation of this result.

(c) Show that the velocity of wave propagation, V, is the same as the shock velocity separating $\rho = \rho_1$ from $\rho = \rho_2$ (occurring if $\nu = 0$).

1.3 Statement: Haberman problem 78.07.

Suppose that a traffic light turned from green to yellow before turning red. How would you mathematically model the yellow light? Note: you do need not solve any problems corresponding to your model for this exercise.

1.4 Statement: TFPa17. Shock interaction with a traffic light.

At time $t = 0$, the traffic pattern on a long highway consists of two sections of constant concentration, joined by a shock which moves in the positive x direction, as shown in figure 1.1. If a traffic light at $x = 0$ turns (at time $t = 0$) and remains red, describe the resultant motion. Let the position of the shock at time $t = 0$ be given by $x = -L < 0$, and assume that $q = \dfrac{4\,q_m}{\rho_j^2}\,\rho\,(\rho_j - \rho)$ — with $0 < \rho_0 < \rho_1 < (1/2)\,\rho_j = \rho_m$.

1.5 Statement: TFPb18. Compute breaking times for IV problems.

The solutions to the initial value problems below are smooth for $-\infty < x < \infty$ and $0 \leq t < t_c$, where t_c is some constant. For $t = t_c$ the partial derivatives of the solution cease to exist at some $x = x_c$ — in fact, they become infinite there. To continue the solution past $t = t_c$, one must introduce a shock — which forms at $x = x_c$ and $t = t_c$.

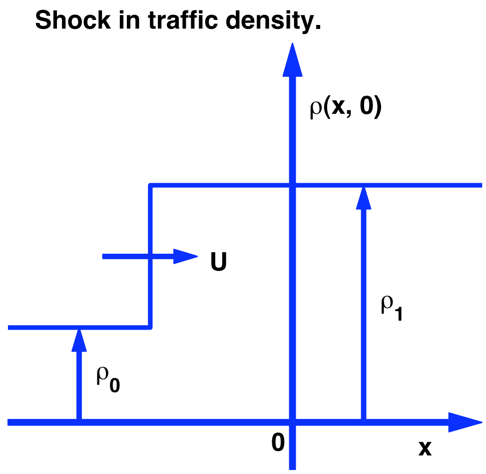

Figure 1.1: TFPa17: Shock in traffic density.

In each case compute t_c, x_c, and ξ_c — where ξ_c is the label for the characteristic along which the first breaking occurs (i.e.: $x = \xi_c$ at $t = 0$ for this characteristic.)

A. Let $\rho(x,\,0) = 1 - \dfrac{1}{1+x^2}$ (for $-\infty < x < \infty$), with

$$\rho_t + \left(\rho\,(1 - \frac{1}{2}\,\rho)\right)_x = 0, \quad \text{for} \quad -\infty < x < \infty \;\text{ and }\; t > 0. \tag{1.3}$$

Note that $q = q(\rho) = \rho\,(1 - \dfrac{1}{2}\,\rho)$ and that $c = c(\rho) = 1 - \rho$.

B. Same as **A**, but change the initial conditions to $\rho(x,\,0) = 1 - \operatorname{sech}(x)$, for $-\infty < x < \infty$.

NOTE: Exact answers are required. Do not resort to calculators till the very end, when you may wish (not required) to evaluate expressions involving quantities such as $\sqrt{3}$ or $\ln(1 + \sqrt{2})$.

2 Special Problems.

2.1 Statement: TFPB20. An initial value problem with a simple quadratic flow.

Solve the following initial value problem **for all times** $t \geq 0$, including shocks if necessary:

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = 0\,, \quad \text{where} \quad -\infty < x < \infty \;\text{ and }\; t > 0\,, \tag{2.4}$$

with

$$u(x,\,0) = \begin{cases} 0 & \text{for } \ x < 0 \ \text{ and } \ x > 1, \\ 1 & \text{for } \ 0 < x < 1. \end{cases} \tag{2.5}$$

When computing the shock speed, note that $\boldsymbol{q = \frac{1}{2}\,u^2}$.
Draw a picture of the solution for several representative times, including $t = 0$ and t very large. Similarly, draw a picture of the characteristics and the shock(s) in space–time.

2.2 Statement: Secants and tangents for quadratic curves.

Here we consider differentiable curves $y = y(x)$ with the following property:

> **The slope of any secant line connecting two points in the curve is equal to the average of the slopes of the curve $\boldsymbol{y = y(x)}$ at the points being connected.** (2.6)

In formulas: for any points $x_1 < x_2$,

$$\frac{y(x_2) - y(x_1)}{x_2 - x_1} = \frac{1}{2}\,\left(y'(x_1) + y'(x_2)\right). \tag{2.7}$$

Show that $\boldsymbol{y = y(x)}$ has this property if, and only if, it is a quadratic polynomial.
Hint. Showing that quadratic polynomials have this property is a simple direct calculation. To show the inverse, write an ode that y must satisfy, and solve it.

THE END.