18.311 — MIT (Spring 2013)

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March 10, 2013.

Problem Set # 03. Due: Mon. March 18.

Turn it in before 3:00 PM, in the box provided in Room 2-285.

IMPORTANT: The Regular and the Special Problems must be **stapled in TWO SEPARATE packages,** each with your **FULL NAME** clearly spelled.

Contents

1 Regular Problems.

1.1 Statement: Haberman problem 63.07.

Consider exercise 61.3. Suppose that the drivers accelerate in such a fashion that

$$\alpha = u_t + u\,u_x = -\frac{a^2}{\rho}\rho_x, \quad \text{where } a > 0 \text{ is a constant.} \tag{1.1}$$

(a) Physically interpret this situation.

(b) If u only depends on ρ, and the equation for conservation of cars is valid, show that

$$\frac{du}{d\rho} = -\frac{a}{\rho}. \tag{1.2}$$

(c) Solve the differential equation in part **(b)**, subject to the condition that $u(\rho_{\mathsf{max}}) = 0$. The resulting flow–density curve fits quite well to the Lincoln Tunnel data.

(d) Show that a is the velocity that corresponds to the road's capacity.

(e) Discuss objections to the theory for small densities.

1.2 Statement: Haberman problem 71.01.

Experiments in the Lincoln Tunnel (combined with theoretical work discussed in exercise 63.7) suggests that the traffic flow function is, approximately,

$$q(\rho) = a\,\rho\,\left(\ln(\rho_{\mathsf{max}}) - \ln(\rho)\right), \tag{1.3}$$

where a and ρ_{max} are known constants. Suppose that the initial density $\rho(x,\,0)$ varies linearly from bumper-to-bumper traffic (behind $x = -x_0 < 0$) to no traffic (ahead of $x = 0$), as sketched in figure 71-6. Two hours later, where does $\rho = \dfrac{1}{2}\,\rho_{\mathsf{max}}$?

1.3 Statement: Haberman problem 73.09.

At which velocity does the information that the traffic light changed from red to green travels?

1.4 Statement: Haberman problem 72.02.

If $u = u_{\mathsf{max}}\,(1 - \rho^2/\rho_{\mathsf{max}}^2)$, then what is the velocity of a traffic shock separating densities ρ_0 and ρ_1? Simplify the expression as much as possible. Show that the shock velocity is **not** (see note below) the average of the density wave velocities associated with ρ_0 and ρ_1.

Note: In fact, show that the only case where the shock velocity is the average of the density wave velocities associated with ρ_0 and ρ_1 is when $\rho_0 = \rho_1$ — and then (of course) there is no shock.

1.5 Statement: Haberman problem 77.05.

Suppose that, instead of $u = U(\rho)$, the car velocity u is

$$u = U(\rho) - \frac{\nu}{\rho}\,\rho_x, \tag{1.4}$$

where ν is a constant.

(a) What sign should $\boldsymbol{\nu}$ have for this expression to be physically reasonable?

(b) What equation now describes conservation of cars?

(c) Assume that $U(\rho) = u_{\mathsf{max}}\,(1 - \rho/\rho_{\mathsf{max}})$. Show that

$$\rho_t + u_{\mathsf{max}}\left(1 - \frac{2\,\rho}{\rho_{\mathsf{max}}}\right)\rho_x = \nu\,\rho_{xx}, \tag{1.5}$$

called **Burger's equation.**

1.6 Statement: Haberman problem 78.01.

Suppose that the initial traffic density is

$$\rho(x,\,0) = \begin{cases} \rho_0 & \text{for } \; x < 0, \\ \rho_1 & \text{for } \; x > 0, \end{cases} \tag{1.6}$$

where ρ_0 and ρ_1 are constants. Consider the two cases, $\rho_0 < \rho_1$, and $\rho_0 > \rho_1$. For which of the preceding cases is a density shock necessary? Briefly explain.

1.7 Statement: Haberman problem 79.02.

Suppose that

$$\rho(x,\,0) = \begin{cases} \rho_0 & \text{for } \; x > 0, \\ 0 & \text{for } \; x < 0. \end{cases} \tag{1.7}$$

Determine the velocity of the shock. Briefly give a physical explanation of the result. What does this shock correspond to?

1.8 Statement: Linear 1st order PDE problem 04.

Discuss the two problems

$$u_x + 2\,x\,u_y = y, \qquad \text{with} \qquad \begin{cases} \text{(a) } u(x,\,x^2) = 1 & \text{for } \; -1 < x < 1, \\ \text{(b) } u(x,\,x^2) = \frac{1}{3}\,x^3 + \pi & \text{for } \; -1 < x < 1. \end{cases} \tag{1.8}$$

How many solutions exist in each case?

Note that the data in these problems is prescribed along a characteristic!

2 Special Problems.

2.1 Statement: Haberman problem 78.08.

Determine the traffic density on a semi–infinity ($x > 0$) highway for which the density at the entrance is

$$\rho(0,\,t) = \begin{cases} \rho_1 & \text{for } \ 0 < t < \tau, \\ \rho_0 & \text{for } \ \tau < t, \end{cases} \tag{2.9}$$

where $\tau > 0$ is constant, and the initial density is uniform along the highway — assume that $\rho(x,\,0) = \rho_0$, for $x > 0$. Furthermore, assume that ρ_1 is lighter traffic than ρ_0, and that both are light traffic; in fact assume that $u(\rho) = u_{\mathsf{max}}\,(1 - \rho/\rho_{\mathsf{max}})$ and that $\rho_1 < \rho_0 < \rho_{\mathsf{max}}/2$. Sketch the density at various values of time.

Hint. Compute the characteristics starting both from the boundary $x = 0$ and $t \geq 0$, as well as those that start at $x \geq 0$ and $t = 0$. Then draw them and look for gaps as well as crossings. Resolve the gaps by inserting expansion fans, and the crossings by inserting shocks. Make sure to find the solution for **all** *times. In particular, in the solution you will find out that there is a shock which starts interacting with a rarefaction fan at some critical location and time $(x_c,\,t_c)$. Compute $(x_c,\,t_c)$, and the shock path* **beyond** *$t = t_c$. Draw the characteristics and shock path after you have resolved all crossings, and filled any gaps by expansion fans. Make sure that the shock satisfies* **both** *the jump and the entropy conditions.*

2.2 Statement: TFPa13. Shock formation time.

Let $\rho(x,\,0) = \left\{ \begin{array}{ll} \rho_0 & \text{for } |x| \geq d\,, \\ \rho_0 + (\rho_1 - \rho_0)(1 - (x/d)^2) & \text{for } |x| \leq d\,, \end{array} \right\} \quad \left\{ \begin{array}{l} \text{where } 0 < \rho_1 < \rho_0 < \rho_j \\ \text{and } d > 0 \text{ is a constant.} \end{array} \right.$

If $q = (4q_m/\rho_j^2)\,\rho\,(\rho_j - \rho)$, find the point $(x_s,\,t_s)$ at which a shock first forms in the traffic flow.

THE END.