18.311 — MIT (Spring 2013)

Rodolfo R. Rosales (MIT, Math. Dept., 2-337, Cambridge, MA 02139).

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Problem Set # 02. Due: Friday March 8.

Turn it in before 3:00 PM, in the box provided in Room 2-285.

IMPORTANT: The Regular and the Special Problems must be **stapled in TWO SEPARATE packages,** each with your **FULL NAME** clearly spelled.

Contents

1 Regular Problems.

1.1 Statement: Haberman problem 58.02.

Assume that the probability P of exactly one car being located in any fixed short segment of highway of length Δx is approximately proportional to the length, $P = \lambda \Delta x$. Also assume that the probability of two or more cars in that segment of highway is negligible.[1]

(a) Show that the probability of there being exactly n vehicles along a highway of length x, $P_n(x)$, satisfies the **Poisson distribution,**

$$P_n(x) = \frac{1}{n!}\,(\lambda\,x)^n\,e^{-\lambda\,x}. \tag{1.1}$$

Hint: *Consider $P_n(x+\Delta x)$, and form a differential equation for $P_n(x)$, (see section 3.6).*

(b) Evaluate and interpret the following quantities

1. $P_n(0)$, $n \neq 0$.
2. $P_0(x)$, $x \neq 0$.
3. $P_1(x)$, $x \neq 0$.

(c) Calculate the expected number of cars on a highway of length x. Interpret your answer.

1.2 Statement: Haberman problem 58.03.

Consider the process of calculating the traffic density ρ, from knowledge of the car positions, as follows:

$$\left.\begin{array}{l}\text{At any given location } x_0 \text{ in space (for a given time } t_0\text{), take the road}\\ \text{interval given by } |x - x_0| \leq 0.5\,\Delta x \text{ (for some ``small'' length } \Delta x\text{) and}\\ \text{count the number of cars } N \text{ in this interval. Then: } \boldsymbol{\rho(x_0,\,t_0) = N/\Delta x}.\end{array}\right\} \tag{1.2}$$

This definition depends on the choice of Δx, but the idea is to take Δx small, but large enough that N is "large". In this case, provided that the number of cars per unit length does not change too fast (neither in space, nor in time), a "good" measurement of the density can be obtained. This exercise will guide you through the verification of this.

NOTE 1: *Since cars are not points, the prescription in (1.2) has an un-resolved issue: What to do with cars which are neither fully inside, nor fully outside the counting interval.* We could spend a lot of energy on this issue, but, for the purposes of this problem, replacing the cars by points (e.g.: the position of their front bumper) is good enough. In addition, when a point is right at the boundary, do not count it. Fancier definitions will not change the basic point this exercise aims to illustrate, but will succeed in making the problem harder, and the point behind it less obvious.

First, notice that the defined density has a discontinuity every time an additional[2] car is included — such discontinuities appear as we either change Δx, or move in space and/or time.

(a) What is the jump $\Delta\rho$ in density (i.e.: the magnitude of the discontinuity)?

[1] Actually, you may assume that these approximations get better and better as $\Delta x \to 0$.

[2] See note 2.

(b) If cars are equally spaced (say: $\rho_c = 100$ per mile), how long must the measuring interval be such that the discontinuities in density are less than 5 percent of the density? Namely: so that $\Delta\rho < \rho/20$. How many cars are then contained in this measuring distance?

(c) Generalize your results to a roadway with a constant density ρ_c per mile.

From item **(c)** a formula of the form $\Delta x > \Delta_c$ will follow — where Δ_c will depend on what size of error (i.e.: the jumps in $\Delta\rho/\rho$) you are willing to tolerate (in this problem 1/20). Let then L be a typical length scale over which the traffic road conditions change. Then the continuum approximation, in which we replace the cars by a density, will make sense as long as $\Delta_c \ll L$.

NOTE 2: It is, in principle, possible that two cars may be added (or dropped) from the counting interval in (1.2) as either Δx is changed, or the counting interval is moved in space, or time passes. But these situations are rare. Therefore, for simplicity, ignore them in your answer to this problem.

1.3 Statement: Haberman problem 60.02.

Suppose that we are interested in the change in the number of cars $N(t)$ between two observers, one fixed at $x = a$, and the other one moving in some prescribed manner, $x = b(t)$:

$$N(t) = \int_a^{b(t)} \rho(x,\, t)dx. \tag{1.3}$$

(a) The derivative of an integral with a variable limit is

$$\frac{dN}{dt} = \frac{db}{dt}\,\rho(b,\, t) + \int_a^{b(t)} \frac{\partial \rho}{\partial t}(x,\, t)\, dx. \tag{1.4}$$

Show this result either by considering $\lim\limits_{\Delta t \to 0} (N(t + \Delta t) - N(t))/\Delta t$, or by using the chain rule for derivatives.

(b) Using $\dfrac{\partial \rho}{\partial t} = -\dfrac{\partial}{\partial x}(\rho\, u)$, show that

$$\frac{dN}{dt} = -\rho(b,\, t)\,\left[u(b,\, t) - \frac{db}{dt}\right] + \rho(a,\, t)\, u(a,\, t). \tag{1.5}$$

(c) Interpret the result of part **(b)** if the moving observer is in a car moving with the traffic.

1.4 Statement: Haberman problems 61.01 & 61.02.

Many state laws say that, for each 10 m.p.h (16 k.p.h) of speed you should stay at least one car length behind the car in front. Assuming that people obey this law "literally" (i.e. they use *exactly* one car length), determine the density of cars as a function of speed (assume that the average length of a car is 16 feet (5 meters)). There is another law that gives a maximum speed limit (assume that this is 50 m.p.h (80 k.p.h)). **Find the flow of cars as a function of density, $q = q(\rho)$.**

The state laws on following distances stated in the prior paragraph, were developed in order to prescribe spacing between cars such that rear-end collisions could be avoided, as follows:

(a) Assume that the car immediately ahead stops instantaneously. How far would the car following at $\boldsymbol{u}$ m.p.h travel, if

  (1) the driver's reaction time was $\boldsymbol{\tau}$, and

  (2) after a $\boldsymbol{\tau}$ delay, the driver decelerated at constant maximum deceleration $\boldsymbol{\alpha}$?

(b) The calculation in part **(a)** may seem somewhat conservative, since cars rarely stop instantaneously. Instead, assume that the first car also decelerates at the same maximum rate $\boldsymbol{\alpha}$, but the driver following still takes time $\boldsymbol{\tau}$ to react. How far back does a car have to be, traveling at $\boldsymbol{u}$ m.p.h, in order to prevent a rear-end collision?

(c) Show that the law described in the first paragraph of this problem corresponds to part **(b)**, if the human reaction time is about 1 second and the length of a car is about 16 feet (5 meters).

Note: *What part* **c** *is asking you to do is to justify/derive the state law prescription, using the calculations in part* **b** *to arrive at a car-to-car separation that will avoid a collision when the cars are forced to brake.*

1.5 Statement: Haberman problem 71.04.

Consider the p.d.e. $\rho_t + q_x = \rho_t + c(\rho)\,\rho_x = 0$, where $q = q(\rho)$ and $c = c(\rho) = \dfrac{dq}{d\rho}(\rho)$. Assume that $\rho = \rho(x,\,t)$ is a smooth solution of the equation, and let $c = c(x,\,t) = c(\rho)$. **Show that:**

$$c_t + c\,c_x = 0. \tag{1.6}$$

1.6 Statement: Linear 1st order PDE problem 01.

Part 1. Find the general solutions to the two 1st order linear scalar PDE

$$x\,u_x + y\,u_y = 0, \qquad \text{and} \qquad y\,v_x - x\,v_y = 0. \tag{1.7}$$

Hint: *The general solutions take a particular simple form in polar coordinates.*

Part 2. For u, find the solution such that on the circle $x^2 + y^2 = 2$, it satisfies $u = x$. *Where is this solution determined by the data given?*

Part 3. Is there a solution to the equation for v such that $v(x, 0) = x$, for $-\infty < x < \infty$?

Part 4. How does the general solution for u changes if the equation is modified to

$$x\, u_x + y\, u_y = (x^2 + y^2)\, \sin(x^2 + y^2)? \tag{1.8}$$

1.7 Statement: Linear 1st order PDE problem 03.

Consider the following problem

$$u_x + 2\, x\, u_y = y, \qquad \text{with} \qquad u(0,\, y) = f(y) \quad \text{for} \quad -\infty < y < \infty, \tag{1.9}$$

where $f = f(y)$ is an "arbitrary" function.

Part 1.
Use the method of characteristics to solve this problem.
Explicitly write the solution $\boldsymbol{u = u(x,\, y)}$ as a function of $\boldsymbol{x}$ and $\boldsymbol{y}$, using $\boldsymbol{f}$.
Hint. *Write the characteristic equations using x as a parameter on them. Then solve these equations using the initial data (for $x = 0$) $y = \tau$ and $u = f(\tau)$, for $-\infty < \tau < \infty$. Finally, eliminate τ, to get u as a function of x and y.*

Part 2.
In which part of the $(x,\, y)$ plane is the solution determined?
Hint. *Draw, in the x-y plane, the characteristic curves computed in part 1.*

Part 3.
Let f have a continuous derivative. Are then the partial derivatives u_x and u_x continuous?

2 Special Problems.

2.1 Statement: Linear 1st order PDE problem 07.

A function $u = u(x,\, y)$ is called *homogeneous of degree* $n > 0$ if and only if $u(\lambda\, x,\, \lambda\, y) = \lambda^n\, u(x,\, y)$, for any constant λ, over the range of independent variables for which u is defined.

Part 1. Consider the homogeneous functions of degree n, defined on the right hand plane $x > 0$, and obtain a pde that all such functions must satisfy. [**Hint:** differentiate the equation satisfied by u, and set $\lambda = 1$.]

Part 2. Use the method of characteristics to find the general solution of the pde derived in part 1, and show that all its solutions are homogeneous functions of degree n.

2.2 Statement: TFPb10. Solve linear equation by characteristics.

Find the solution to the equation

$$x\,\phi_y + \phi_x = \phi\,, \tag{2.10}$$

with $\phi = y$ for $x = 0$ and $y > 0$. **Describe and plot the region in space where the characteristics reach** (thus where the solution is defined.)

THE END.