# 18.311 — MIT (Spring 2013)

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## Problem Set # 01. Due: Monday February 25.

**Turn it in before 3:00 PM, in the box provided in Room 2-285.**

**IMPORTANT:** The Regular and the Special Problems must be **stapled in TWO SEPARATE packages,** each with your **FULL NAME** clearly spelled.

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# 1 Regular Problems.

## 1.1 ExID02 statement: Single variable dependence implicit differentiation.

In each case compute $y' = \dfrac{dy}{dx}$ as a function of $y$ and $x$, given that $y = y(x)$ satisfies:

1. $x^3\,y + 2\,x + 3\,y^3 = 0$.
2. $y = \cos(x + y^2)$.
3. $\ln(1 + y) = e^x$.
4. $\cos^2(y^2) = e^{-x^2}$.
5. $y = f(1 + y\,x)$.
6. $1 = y\,f(x + y)$.

Note: In (5) and (6) $f$ is an arbitrary function.

## 1.2 ExID11 statement: Two variable dependence implicit differentiation.

In each case compute $u_x = \dfrac{\partial u}{\partial x}$ and $u_y = \dfrac{\partial u}{\partial y}$ (as functions of $u$, $x$, and $y$), given that $u = u(x,\,y)$ satisfies:

1. $x^3 y + 2xu + yu^3 = 0$. 2. $y = f(y + ux)$. 3. $\ln(1+y) = u e^{xu}$.

Note: In (2) $f$ is an arbitrary function of a single variable, $f = f(\zeta)$. Assume that $f' \neq 0$.

### 1.3 ExPD05 statement: Check 1D kinematic equation's implicit solution.

By direct substitution, show that

$$u = f(x - V(u)\,t), \tag{1.1}$$

(where $f$ is an arbitrary function and $V = V(u)$ is a given function) is a solution of the equation:

$$u_t + V(u)\,u_x = 0. \tag{1.2}$$

Notice that:

1. Equation (1.1) **defines $u$ implicitly,** so you will have to find $u_t$ and $u_x$ by implicit differentiation.
2. Equation (1.2) is nonlinear, except for the trivial case when $V$ is a constant function — in which case this problem reduces to the problem in ExPD03.

## 2 Special Problems.

### 2.1 DiAn02 statement: Speed of a non-linear diffusion front.

Consider some substance diffusing in an **isotropic**[1] **medium at rest.** Then, as shown in the lectures, the following equation applies

$$\mathcal{C}_t = \operatorname{div}(\nu\,\nabla\mathcal{C}), \tag{2.3}$$

where

**1.** Fick's law of diffusion was assumed: the substance flux due to diffusion is along the gradient of the concentration, from higher to lower concentrations. For this it is important that the medium be isotropic — else the flux diffusion may occur along directions that depend both on the gradient of the concentration, as well as special directions in the media.
**2.** $\mathcal{C} = \mathcal{C}(\vec{x},\, t)$ is the substance concentration (mass per volume) — e.g.: grams per liter.
**3.** $\nu > 0$ is the *diffusion coefficient.*
**4.** $\nabla\mathcal{C}$ is the gradient of the concentration, and div denotes the divergence of a vector field.

[1] **Isotropic** means that the properties of the medium are invariant under rotation.

*When $\nu$ is a constant* (2.3) reduces to the *linear diffusion equation*

$$\mathcal{C}_t = \nu\,\Delta\,\mathcal{C}, \tag{2.4}$$

where $\Delta$ is the *Laplacian operator*, $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$.

However, there are situations where the diffusion coefficient is **not** constant, and depends (for example) on the concentration itself: $\nu = \nu(\mathcal{C})$. Here we will consider the particular case where $\nu \propto \mathcal{C}^2$ and the medium is homogeneous.[2] Then (2.3) reduces to

$$\mathcal{C}_t = \mu\,\mathrm{div}\left(\mathcal{C}^2\,\nabla\mathcal{C}\right), \tag{2.5}$$

where $\boldsymbol{\mu > 0}$ **is a constant.**

Under conditions where (2.5) applies, imagine that at $t = 0$ there is a very tiny blob of substance somewhere in the media. Then, due to the diffusion, the size of the blob will increase with time — with a sharp edge between the region where $\mathcal{C} > 0$, and the region where there is no substance. *Note: In this the behavior of (2.5) differs from that of (2.4). In the case of the linear diffusion equation, the blob's edge ceases to be sharp for $t > 0$, even if the initial blob has a sharp edge.*

## Problem Tasks. What you are expected to do.

Let $\boldsymbol{M}$ **be the total mass in the blob,** and make the approximation that, **at time $\boldsymbol{t = 0}$ the blob is just a point** — i.e.: all the substance's mass is concentrated in a blob so tiny that you can think of it as a point. Then perform the tasks below, ***using qualitative (but precise) and physical arguments only*** — *do not solve the equation.*

**q1.** What dimensions does $\mu$ have?

**q2.** Argue that the shape of the blob is a sphere for $t > 0$.

**q3.** Find a formula for the radius of the blob $R = R(t)$ as a function of time — namely: a formula of the form $R = \alpha\, f(t)$, where $\alpha$ is a dimension-less constant (a number), and $f(t)$ is some function of time. You will not able to calculate $\alpha$ without solving the p.d.e. (2.5), which you are not expected to do. But you should be able to fully determine the function $f(t)$.

Assume now that $\boldsymbol{R(t) = 5}$ **mm** when $\boldsymbol{t = 3.75}$ **sec.**

**q4.** What value does $R$ take for $t = 960\,\mathrm{sec} = 16\,\mathrm{min}$?

**q5.** For what value of $t$ is $R = 5\,\mathrm{cm}$?

[2] **Homogeneous** means that the properties of the medium are the same everywhere.

**q6.** Would your answers change if the nonlinear diffusion occurred in the plane, instead of in 3-D? In particular, what are the answers to **q1** and **q3** in 2-D?

*Hint. The formula giving the radius as a function of time must involve physical constants that allow it to transform time into length. These physical constants must be, in turn, result from the physical constants in the problem, and only them — e.g.: if you write a formula that involves the speed of light in it, something went wrong with your reasoning!*

## 2.2 Statement: Compute a channel flow rate function.

It was shown in the lectures that for a river (or a man-made channel) in the plains, under conditions that are not changing too rapidly (quasi-equilibrium), the following equation should apply

$$A_t + q_x = 0, \tag{2.6}$$

where $A = A(x, t)$ is the cross-sectional filled area of the river bed, $x$ measures length along the river, and $q = Q(A)$ is a function giving the flow rate at any point.

That the flow rate $q$ should be a function of $A$ only[3] follows from the assumption of quasi-equilibrium. Then $q$ is determined by a local balance between the friction forces and the force of gravity down the river bed.

Assume now a man-made channel, with uniform triangular cross-section and a uniform (small) downward slope, characterized by an angle $\theta$. Assume also that the frictional forces are proportional to the product of the flow velocity $u$ down the channel, and the wetted perimeter $P_w$ of the channel bed $F_f = C_f\, u\, P_w$. **Derive the form that the flow function $Q$ should have.**

**Hints:** (1) $Q = u\,A$, *where $u$ is determined by the balance of the frictional forces and gravity.* (2) *The wetted perimeter $P_w$ is proportional to some power of $A$.*

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**THE END.**

[3]Possibly also $x$ — i.e. $q = Q(x,\, A)$ — to account for non-uniformities along the river.