Various lecture notes for 18311.

R. R. Rosales (MIT, Math. Dept., 2-337)

April 12, 2013

Abstract

Notes, both complete and/or incomplete, for MIT's 18.311 (Principles of Applied Mathematics). These notes will be updated from time to time. Check the date and version.

Contents

1 Isentropic Gas Dynamics. **1**

1.0.1 Problems. 3

Problem: pressure convexity condition equivalence. 3

Problem: equivalence of the Lagrangian and Eulerian forms. 3

Problem: Lagrangian property of σ. 3

Problem: explicit solution for σ. 3

1.1 Characteristic form of the equations. 3

1.1.1 Simple waves. 4

1.1.2 Problems. 5

Problem: simple wave on the left characteristics. 5

1.2 Shock conditions. 5

1.2.1 Graphical interpretation. 7

1.2.2 Entropy conditions. 8

1.2.3 Problems. 9

Problem: shock conditions when sources are present. 9

List of Figures

1.2.1 Shock Hugoniot diagram for isentropic gas dynamics. 7

1 Isentropic Gas Dynamics.

This section contains notes on the theory of characteristics, and shocks, for the isentropic Euler equations of gas dynamics in 1-D. In the absence of body forces, the conservation form of the equations is

$$\rho_t + (\rho\, u)_x = 0 \quad \text{(conservation of mass)}, \tag{1.1}$$

$$(\rho\, u)_t + (\rho\, u^2 + p)_x = 0 \quad \text{(conservation of momentum)}, \tag{1.2}$$

where $\boldsymbol{\rho}$ is the **mass density**, $\boldsymbol{u}$ is the **flow velocity**, $\boldsymbol{\rho\, u}$ is the **momentum density**, and $\boldsymbol{p}$ is the **pressure**. The pressure is given by an **equation of state**[1]

$$p = p(\rho), \quad \text{where we assume that:} \quad a^2 = \frac{dp}{d\rho} > 0 \;\text{ and }\; \frac{d^2}{d\rho^2}(\rho\, p) > 0. \tag{1.3}$$

The first condition on p is intuitively clear: the pressure should increase as the gas is compressed (ρ growing) — as we will see in § 1.1, $\boldsymbol{a > 0}$ **is the speed of sound.** The reason for the second condition will be clarified when we look at shocks for these equations, in § 1.2. Note, however, that *these conditions are equivalent to*

$$\frac{dp}{d\mathrm{v}} < 0 \;\text{ and }\; \frac{d^2p}{d\mathrm{v}^2} > 0, \tag{1.4}$$

where $\mathrm{v} = 1/\rho$ is the **specific volume** (volume per unit mass). In other words *the pressure is a decreasing, convex, function of the specific volume.*

$$\textbf{Example, polytropic gas:}^2 \quad p = \kappa\, \rho^\gamma = \kappa\, \mathrm{v}^{-\gamma}, \tag{1.5}$$

where κ, $\gamma > 0$ are constants (usually $1 < \gamma < 2$).

It is sometimes convenient to write the equations in **Lagrangian variables** $\boldsymbol{(\sigma,\, t)}$, instead of the **Eulerian variables** $\boldsymbol{(x,\, t)}$ used in (1.1–1.2). Here $\sigma = \sigma(x,\, t)$ is defined by the equations

$$\sigma_t = -\rho\, u \quad \text{and} \quad \sigma_x = \rho, \tag{1.6}$$

which are consistent because of (1.1). In terms of these coordinates the equations take the simple form

$$\mathrm{v}_t - u_\sigma = 0 \quad \text{and} \quad u_t + p_\sigma = 0. \tag{1.7}$$

Note: σ is called a Lagrangian variable because[3]

$$\boldsymbol{\sigma} \textbf{ is constant along the particle paths } \boldsymbol{\dot{x} = u(x,\, t)}. \tag{1.8}$$

Using this fact we can give a physical interpretation to the equations in (1.7), as follows: Take any fixed constant $\sigma_1 < \sigma_2$. These correspond to two particles with paths $x = x_i(t)$ and velocities $u_i = u(x_i(t), t)$. Then

1. $$\int_{\sigma_1}^{\sigma_2} \mathrm{v}\, d\sigma = \int_{x_1}^{x_2} dx = x_2 - x_1 \quad \text{and} \quad \int_{\sigma_1}^{\sigma_2} u_\sigma\, d\sigma = u_1 - u_2, \tag{1.9}$$

where we have used that $d\sigma = \rho\, dx$ at any fixed time. Thus the first equation in (1.7) expresses the evolution of the distance between particles — *conservation of volume.*

2. $$\int_{\sigma_1}^{\sigma_2} u\, d\sigma = \int_{x_1}^{x_2} \rho\, u\, dx \quad \text{and} \quad \int_{\sigma_1}^{\sigma_2} p_\sigma\, d\sigma = p_1 - p_2. \tag{1.10}$$

Thus the second equation in (1.7) is Newton's law applied to the mass of gas between σ_1 and σ_2 — *conservation of momentum.*

---

[1] In general, the pressure depends both on the density and the entropy: $p = p(\rho, S)$.

[2] A polytropic gas is an ideal gas with constant specific heats.

[3] The equations in (1.6) define σ up to a constant. This constant can be selected as follows: Pick a gas particle (any particle), and a constant σ_*. Then set $\sigma = \sigma_*$ along the selected particle path.

1.0.1 Problems.

Problem 1.1 **Pressure convexity condition equivalence.**
Show the equivalence between the conditions on the pressure in (1.3) and those in (1.4).

Problem 1.2 **Equivalence of the Lagrangian and Eulerian forms.** *Show the equivalence between (1.1–1.2) and (1.7).* Hint: write the equations expressing Eulerian derivatives in terms of Lagrangian derivatives. Use this in $\rho_t + (\rho\, u)_x = 0$ and $u_t + u\, u_x + \frac{1}{\rho}\, p_x = 0$ — equivalent to (1.1–1.2).

Problem 1.3 **Lagrangian property of $\boldsymbol{\sigma}$.** *Show that (1.8) applies.* Hint: compute $\dot{\sigma}$ along particle paths.

Problem 1.4 **Explicit solution for $\boldsymbol{\sigma}$.** *Consider a situation of a pipe with a closed end on the left. Then the equations in (1.1–1.2) apply for $x > 0$, with the boundary condition $u = 0$ at $x = 0$. Show that*

$$\sigma = \int_0^x \rho(\tilde{x},\, t)\, d\tilde{x} \quad \textit{solves (1.6).} \tag{1.11}$$

Solutions to the problems.

1.1 Characteristic form of the equations.

Using the first equation to eliminate the time derivative of the density in the second equation, the equations in (1.1–1.1) can be written in the form

$$\rho_t + (\rho\, u)_x = 0 \quad \text{and} \quad u_t + u\, u_x + \frac{a^2}{\rho}\, \rho_x = 0. \tag{1.1.1}$$

This has the vector form $Y_t + A\, Y_x = 0$, where

$$Y = \begin{pmatrix} \rho \\ u \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} u & \rho \\ \frac{a^2}{\rho} & u \end{pmatrix}. \tag{1.1.2}$$

The matrix A has the eigenvalues $\lambda = u \pm a$, with corresponding left-eigenvectors $\ell = (\pm\frac{a}{\rho},\, 1)$. Thus, left-multiplying the equations by the eigenvectors gives

$$u_t + (u \pm a)\, u_x \pm \frac{a}{\rho}\, (\rho_t + (u \pm a)\, \rho_x) = 0. \tag{1.1.3}$$

That is

$$\frac{du}{dt} \pm \frac{a}{\rho}\, \frac{d\rho}{dt} = 0 \quad \text{along} \quad \frac{dx}{dt} = u \pm a. \tag{1.1.4}$$

These are the **characteristic equations.** In particular, define $h = h(\rho)$ by

$$h = \int_0^\rho \frac{a(s)}{s}\, ds. \qquad \text{Example. For a polytropic gas:}^4 \;\; h = \frac{2}{\gamma - 1}\, a. \tag{1.1.5}$$

Then (1.1.4) yields

$$u \pm h \quad \text{is constant along} \quad \frac{dx}{dt} = u \pm a. \tag{1.1.6}$$

$\boldsymbol{u+h}$ is called the **right Riemann invariant**, and $\boldsymbol{u-h}$ is called the **left Riemann invariant**.

Note that, unlike the case in traffic flow, here there are two sets of characteristics, whose equations are coupled. Hence, in general, one cannot solve the characteristic equations as a system of ode's. An exception to this is given in § 1.1.1.

1.1.1 Simple waves.

Consider a situation where the left Riemann invariant is a constant for the initial conditions. Then, because the left Riemann invariant is a constant along each left characteristic

$$u - h = U = \text{constant, everywhere.} \tag{1.1.7}$$

This allows us to write ρ as a function of u everywhere. For example, for a polytropic gas we can write

$$a = \frac{\gamma - 1}{2}\,(u - U). \tag{1.1.8}$$

Substituting this into the equation for the right Riemann invariant then gives

$$2\,u - U \quad \text{is constant along} \quad \frac{dx}{dt} = \frac{\gamma+1}{2}\,u - \frac{\gamma-1}{2}\,U. \tag{1.1.9}$$

In other words, u satisfies the equation

$$u_t + \left(\frac{\gamma+1}{2}\,u - \frac{\gamma-1}{2}\,U\right)\,u_x = 0. \tag{1.1.10}$$

That is, *the problem reduces to the same type of equations that occur in traffic flow, where solutions can be obtained by solving ode.*

The solutions to (1.1.10) exhibit wave steepening, which, in general, leads to multiple values (and infinite slopes). Once infinite slopes develop, the equations become invalid. Hence **the equations in (1.1–1.1) present the same type of difficulty that arises for traffic flow.** In this case, again, it turns out that the correct physical resolution of the difficulty is through the introduction of solutions with discontinuities (**shocks**). See § 1.2.

---

[4] Then $p = \kappa\,\rho^\gamma$.

1.1.2 Problems.

Problem 1.1.1 **Simple wave on the left characteristics.** *How does equation (1.1.10) change when the right Riemann invariant is a constant for the initial conditions? That is, when (1.1.7) is replaced by*

$$u + h = U = \textit{constant, everywhere.} \tag{1.1.11}$$

1.2 Shock conditions.

As shown in § 1.1.1, the Euler equations of Gas Dynamics exhibit wave steepening, which leads to infinite derivatives in a finite time, at which point the equations become invalid . . . unless an appropriate correction to the mathematical model is introduced. As in the case of flood waves and traffic flow, the correct physics is captured by introducing shocks into the solutions. **Shock equations are then needed.** At shocks the differential form of the equations cannot be used, and one must go back to the conservation laws to obtain the shock equations: the **Rankine-Hugoniot jump conditions.** In addition, **"entropy" conditions** are also needed.[5]

The Rankine-Hugoniot jump conditions follow from the conservation of mass and momentum equations in (1.1–1.2) — see remark 1.2.1.

$$-s\,[\rho] + [\rho\, u] = 0 \quad \text{and} \quad -s\,[\rho\, u] + [\rho\, u^2 + p] = 0, \tag{1.2.1}$$

where s is the shock speed and the brackets indicate the jump in the enclosed variable across the shock discontinuity (e.g.: value on the right minus value on the left).

Remark 1.2.1 *Let $\tilde{\rho}$ be the density of a conserved quantity, and let $\tilde{q}$ be the corresponding flux.*[6] *Then a shock must satisfy*

$$-s\,[\tilde{\rho}] + [\tilde{q}] = 0. \tag{1.2.2}$$

Proof. Let the shock be at $x = \xi(t)$, with speed $s = \dot{\xi}$ — we assume that *the shock path is continuous and has a derivative.* For any small time interval $t_1 < t < t_2$, constants a and b can be selected so that $a < \xi < b$. Then conservation tells us that

$$\frac{d}{dt}\left(\int_a^{\xi} \tilde{\rho}\, dx + \int_{\xi}^{b} \tilde{\rho}\, dx\right) = \tilde{q}(a,\, t) - \tilde{q}(b,\, t). \tag{1.2.3}$$

Furthermore, away from the shock (in particular: for $a < x < \xi$ and $\xi < x < b$) it should be that

$$\tilde{\rho}_t + \tilde{q}_x = 0. \tag{1.2.4}$$

[5] Shocks are used only when needed. Shocks should not be added at places where a continuous solution will do.

[6] For example: $\tilde{\rho} = \rho\, u$ and $\tilde{q} = \rho\, u^2 + p$ in gas dynamics.

It follows that (the subscripts L and R indicate evaluation immediately to the left and right of the shock)

$$\begin{aligned}
\tilde{q}(a,\,t)-\tilde{q}(b,\,t) &= s\,\tilde{\rho}_L+\int_a^{\xi}\tilde{\rho}_t\,dx-s\,\tilde{\rho}_R+\int_{\xi}^{b}\tilde{\rho}_t\,dx \;=\; \tilde{\rho}_L-\int_a^{\xi}\tilde{q}_x\,dx-\tilde{\rho}_R-\int_{\xi}^{b}\tilde{q}_x\,dx \\
&= s\,\tilde{\rho}_L-(\tilde{q}_L-\tilde{q}(a,\,t))-s\,\tilde{\rho}_R-(\tilde{q}(b,\,t)-\tilde{q}_R) \\
\Longrightarrow\quad 0 &= s\,(\tilde{\rho}_L-\tilde{\rho}_R)-(\tilde{q}_L-\tilde{q}_R),
\end{aligned} \tag{1.2.5}$$

which is precisely (1.2.2). ♣

We now ***recast (1.2.1) in a form that makes their meaning more clear.***

Let us indicate the values of the variables on each side of the shock by the subscripts 0 and 1, and select the subscripts so that $\boldsymbol{\rho_0<\rho_1}$ — i.e. $\mathbf{v_0>v_1}$.[7] Then eliminate s and u_1 in favor of D and U defined below. This yields[8]

$$\rho_1\,U=D\,(\rho_1-\rho_0)\quad\text{and}\quad D\,(\rho_1-\rho_0)\,u_0-\rho_1\,u_0\,U+D\,\rho_1\,U-\rho_1\,U^2+p_0-p_1=0, \tag{1.2.6}$$

where $\boldsymbol{D=\ s-u_0}$ **is the shock velocity relative to the fluid on the "0" side of the shock,** and $\boldsymbol{U=u_1-u_0}$**.** Finally, using the first equation to eliminate U from the second one, we write the equations in the form.

$$U \;=\; D\,(\rho_1-\rho_0)\,\mathrm{v}_1=\left(1-\frac{\rho_0}{\rho_1}\right)\,D\quad\text{and} \tag{1.2.7}$$

$$\text{either}\qquad D^2 \;=\; \frac{\rho_1}{\rho_0}\,\frac{p_1-p_0}{\rho_1-\rho_0}, \tag{1.2.8}$$

$$\text{or}\qquad D^2\,\rho_0^2 \;=\; -\frac{p_1-p_0}{\mathrm{v}_1-\mathrm{v}_0}. \tag{1.2.9}$$

1. The equations allow for two "branches" of solutions, one with $D>0$, and another with $D<0$.

2. As the shock vanishes, $\rho_1-\rho_0\to 0$, D approaches the sound speed, $D^2\to a^2$. (1.2.10)

3. *D and U have the same sign, with $|U|<|D|$.*
Hence $\boldsymbol{s-u_0=D}$ **and** $\boldsymbol{s-u_1=D-U}$ **have the same sign.** Thus one of these applies:
— The shock moves to the right relative to the flow, $D>0$. A **right shock**.
— The shock moves to the left relative to the flow, $D<0$. A **left shock**. (1.2.11)
We associate the right shocks with the right characteristics, and the left shocks with the left characteristics. As we will show later in § 1.2.2, *the right shocks prevent the right characteristics from crossings, and the left shocks prevent the left characteristics from crossings.*

4. $\rho_0\,D=\rho_0\,(s-u_0)=\rho_1\,(s-u_1)$ is the mass flow through the shock. (1.2.12)
In particular: **for right shocks the mass flows from right to left across the shock**,
while **for left shocks the mass flows from left to right across the shock.**

---

[7] Recall that $\mathrm{v}=1/\rho$ is the specific volume.
[8] Note that $\boldsymbol{p_j=p(\rho_j)=p(\mathrm{v}_j)}$, with $\boldsymbol{p_1>p_0}$.

1.2.1 Graphical interpretation.

Equation (1.2.9) has a nice graphical interpretation. It says that the **shock states** on both sides of the shock, that is: $(\mathrm{v}_0,\, p_0)$ and $(\mathrm{v}_1,\, p_1)$, are **determined by the intersection of two curves** (see figure 1.2.1)

a. The pressure-specific volume (or Hugoniot) curve $p = p(\mathrm{v})$. Notice that this *curve is decreasing and convex* — see (1.4).

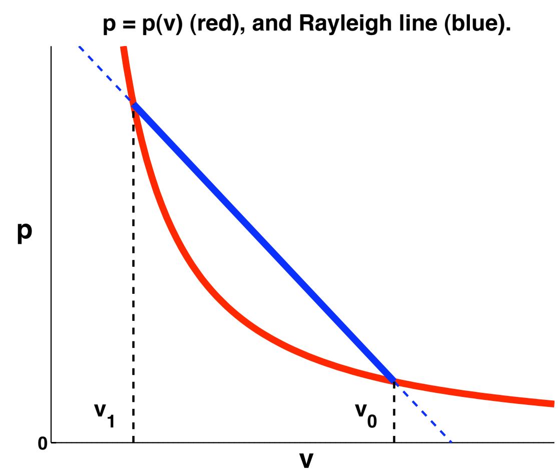
Note: For isentropic Gas Dynamics the fact that the shock states should be on the curve $p = p(\mathrm{v})$, which is convex and decreasing, is a somewhat trivial fact. Thus it may seem puzzling that this curve is given a name. However, this is also true for the complete set of gas dynamic equations (no constant entropy assumption), where what the curve $p = p(\mathrm{v})$ should be is less obvious.

b. A straight line (the Rayleigh line), whose slope is the negative of the square of the mass flux through the shock — see (1.2.12). Because the Hugoniot curve is convex, the Rayleigh line cannot intersect it at any place other than $(\mathrm{v}_0,\, p_0)$ and $(\mathrm{v}_1,\, p_1)$.

The Rayleigh line is the secant line connecting the two shock states.

Note: The Rayleigh line can also be tangent to the Hugoniot curve — i.e.: just a single common point. This corresponds to the limit $\mathrm{v}_0 - \mathrm{v}_1 \to 0$, that is: the characteristics — see (1.2.10).

Figure 1.2.1: Hugoniot diagram for shocks in isentropic gas dynamics. The shock states occur at the intersection of the Hugoniot curve $p = p(\mathrm{v})$ and the Rayleigh line (straight line with slope $-\rho_0^2\, D^2$ joining the shock states). For a right shock, $D > 0$, the left state is $(\mathrm{v}_1,\, p_1)$ and the right state is $(\mathrm{v}_0,\, p_0)$ — "entropy" condition. The reverse occurs for a left shock.



Note the analogy between this situation and the one in Traffic Flow, where the shocks are associated with the intersections between a concave curve — the flow rate $q = Q(\rho)$ — and a straight line, with the intersections being the shock states and tangency corresponding to the limit where shocks become characteristics. Furthermore: in Traffic Flow the fact that $q = Q(\rho)$ is concave plays an important role in relationship with the "entropy" condition. Here the convexity of $p = p(\mathrm{v})$ plays a similar role, as we will see in § 1.2.2.

1.2.2 Entropy conditions.

Not all the solutions to the Rankine-Hugoniot jump conditions (1.7–1.9) are acceptable. Shocks should be introduced only when needed to prevent characteristics from crossing.[9] This means that we may need shocks to prevent the right-characteristics from crossing (this will be the right shocks), or we may need shocks to prevent the left-characteristics from crossing (this will be the left shocks).

Right shocks and right characteristics move to the right relative to the fluid (speed greater than u) — see (1.2.11). Similarly left shocks and left characteristics move to the left relative to the fluid. Thus only right shocks can prevent right characteristics from crossing (characteristics ending at the shock), and only left shocks can prevent left characteristics from crossing. This leads to the following **"entropy conditions"**

$$\left.\begin{array}{ll}\textbf{For right shocks:} & \boldsymbol{u_L + a_L > s > u_R + a_R,} \\ \textbf{For left shocks:} & \boldsymbol{u_L - a_L > s > u_R - a_R,}\end{array}\right\} \tag{1.2.13}$$

where the subscripts L and R indicate the states immediately to the left and right of the shock, respectively. Now, from the convexity of the Hugoniot curve, it follows that (see figure 1.2.1)

$$\rho_0^2\, a_0^2 < \rho_0^2\, D^2 = \rho_0^2\, (s - u_0)^2 \quad \text{and} \quad \rho_1^2\, a_1^2 > \rho_0^2\, D^2 = \rho_1^2\, (s - u_1)^2, \tag{1.2.14}$$

where we have used (1.2.12), and the fact that $\rho^2\, a^2 = -\frac{dp}{d\mathrm{v}}$. For a right shock $s - u_0$ and $s - u_1$ are both positive, so that $a_0 < s - u_0$ and $a_1 > s - u_1$. Similarly, for a left shock $a_0 < u_0 - s$ and $a_1 > u_1 - s$. We conclude that

$$\left.\begin{array}{ll}\textbf{For right shocks:} & \boldsymbol{u_1 + a_1 > s > u_0 + a_0.} \\ \textbf{For left shocks:} & \boldsymbol{u_0 - a_0 > s > u_1 - a_1.}\end{array}\right\} \tag{1.2.15}$$

Comparing this with (1.2.13) we see that **the "entropy" condition is satisfied if and only if**

$$\left.\begin{array}{llll}\textbf{For right shocks:} & \boldsymbol{(u_L,\, \mathrm{v}_L) = (u_1,\, \mathrm{v}_1)} & \textbf{and} & \boldsymbol{(u_R,\, \mathrm{v}_R) = (u_0,\, \mathrm{v}_0).} \\ \textbf{For left shocks:} & \boldsymbol{(u_L,\, \mathrm{v}_L) = (u_0,\, \mathrm{v}_0)} & \textbf{and} & \boldsymbol{(u_R,\, \mathrm{v}_R) = (u_1,\, \mathrm{v}_1).}\end{array}\right\} \tag{1.2.16}$$

From (1.2.12) it then follows that:

The "entropy" condition is satisfied if and only if the shocks are compressive. (1.2.17)

That is: as the gas goes through the shock, it is compressed (density and pressure go up). Similarly, since D and U have the same sign

The "entropy" condition is satisfied if and only if the right shocks accelerate the fluid to the right, while the left shocks accelerate the fluid to the left. Hence, in both cases, $\boldsymbol{u_L > u_R}$. (1.2.18)

[9] **Failure to do this** not only **introduces non-physical features** into the solution, but in fact makes the mathematical problem **ill-posed.** Uniqueness is lost, meaning that, given the initial data and appropriate boundary values, more than one solution (in fact, infinitely many) are possible if shocks not satisfying the "entropy" conditions are allowed.

Note that a shock that violates the "entropy" condition would expand the fluid as it went through it, while decelerating the fluid in the direction of propagation. Such shocks are not observed in nature and, as pointed out earlier, would lead to a very badly behaved mathematical model as well.

Remark 1.2.2 **The arrow of time.** *In the absence of shocks, the equations in (1.1–1.2) are reversible. This means that if* $(\rho(x,\,t),\,u(x,\,t))$ *is a solution, then* $(\rho(x,\,-t),\,-u(x,\,-t))$ *is also a solution. However, once shocks appear, reversibility is lost. This because, under the transformation* $(\rho(x,\,t),\,u(x,\,t)) \rightarrow (\rho(x,\,-t),\,-u(x,\,-t))$*, a shock that satisfies the "entropy" condition is mapped into one that does not. Just as in the case of traffic flow,* shocks produce information loss.

Remark 1.2.3 **Is isentropic flow consistent with shocks?** *The equations in (1.1–1.2) are derived under the assumption of an adiabatic flow (i.e.: very slow changes). Hence one may question their applicability when shocks arise. When the shocks are not sufficiently weak, this questioning is justified, and one must revert to the full set of Euler equations (including conservation of energy) to deal with the situation. However, when the shocks are weak, (1.1–1.2) is still a reasonable approximation. This is because the amount of entropy created by weak shocks is "high order" — it can be shown that it is roughly proportional to the cube of the shock strength.*

A non-dimensional measure of the shock strength is provided by $M - 1$*, where* M *is the* **Mach number***, defined by* $\boldsymbol{M = |D|/a_0}$*. The* **"entropy" condition guarantees that** $\boldsymbol{M > 1}$**, in fact, it is equivalent to it.**

1.2.3 Problems.

Problem 1.2.1 **Shock conditions when sources are present.** *How are the Rankine-Hugoniot jump conditions in (1.2.2) affected by the presence of sources?* Hint. In this case (1.2.2–1.2.4) must be replaced by

$$\frac{d}{dt}\left(\int_a^\xi \tilde{\rho}\,dx + \int_\xi^b \tilde{\rho}\,dx\right) = \tilde{q}(a,\,t) - \tilde{q}(b,\,t) + \int_a^b S\,dx \quad \text{and} \quad \tilde{\rho}_t + \tilde{q}_x = S, \tag{1.2.19}$$

where $S = S(x,\,t)$ — or, possibly, $S = S(x,\,t,\,\tilde{\rho})$ — is the source term.

The End.