18.311 — MIT (Spring 2012)

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May 11, 2012.

Problem Set # 06.

Due: Last day of lectures.

Turn it in before 3:00 PM, in the box provided in Room 2-108.

IMPORTANT: The Regular and the Special Problems must be **stapled in TWO SEPARATE packages,** each with your **FULL NAME** clearly spelled.

Contents

1 Regular Problems.

Points: §**1.1, 15 points.** §**1.2, 15 points.** §**1.3, 15 points.**

1.1 vNSA01 Scheme A. Forward differences for $u_t + u_x = 0$.

Do problem *"vNSA01 scheme A"* from the "vNSA von Neumann Stability Analysis Problems" series.

1.2 vNSA04 Scheme D. Centered differences for $u_t = u_{xx}$.

Do problem *"vNSA04 scheme D"* from the "vNSA von Neumann Stability Analysis Problems" series.

1.3 ExPD21 statement:
Separation of variables for the Laplace equation in a rectangle.

Consider the question of determining the steady state temperature in a thin rectangular plate such that: (a) The temperature is prescribed at the edges of the plate, and (b) The facets of the plate (top and bottom) plate are insulated. The solution to this problem can be written as the sum of the solutions to four problems of the form[1]

$$T_{xx} + T_{yy} = 0, \quad \text{for } \ 0 \le x \le \pi \quad \text{and } \ 0 \le y \le L, \tag{1.1}$$

with boundary conditions $\qquad T(0,\, y) = T(\pi,\, y) = T(x,\, 0) = 0 \quad \text{and} \quad T(x,\, L) = h(x), \qquad (1.2)$
where $\boldsymbol{L > 0}$ is a constant, and $\boldsymbol{h = h(x)}$ is some prescribed function. In turn, *the solution to (1.1 – 1.2) can be written as a (possibly infinite) sum of separation of variables solutions.*

Separation of variables solutions to (1.1 – 1.2) are product solutions of the form

$$T(x,\, y) = \phi(x)\, \psi(y), \tag{1.3}$$

where $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are some functions. In particular, from (1.2), it must be that

$$\phi(0) = \phi(\pi) = \psi(0) = 0 \quad \text{and} \ \ h(x) = \phi(x)\, \psi(L). \tag{1.4}$$

However, for the separated solutions, the function $\boldsymbol{h}$ **is NOT prescribed:** It is whatever the form in (1.3) allows. The general h is obtained by adding separation of variables solutions.

These are the tasks in this exercise:

1. **Find ALL the separation of variables solutions.**
2. In exercise ExPD04 we stated that all the solutions to the Laplace equation — i.e.: (1.1) — must have the form $\boldsymbol{T = f(x - i\,y) + g(x + i\,y)}$, for some functions $\boldsymbol{f}$ and $\boldsymbol{g}$. *Note that these two functions must make sense for complex arguments, and have derivatives in terms of these complex arguments.* — which means that f and g must be analytic functions.

   Show that the result in the paragraph above applies to all the separated solutions found in item 1. Namely: find the functions f and g that correspond to each separated solution.

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[1] Here we use non-dimensional variables, selected so that one side of the plate has length π.

2 Special Problems.

Points: § **2.1, 20 points.** § **2.2, 20 points.** § **2.3, 20 points.** § **2.4, 20 points.**

2.1 vNSA03 Scheme C. Centered differences for $u_t + u_x = 0$.

Do problem *"vNSA03 scheme C"* from the "vNSA von Neumann Stability Analysis Problems" series.

2.2 vNSA07 Scheme G. Implicit centered differences for $u_t = u_{xx}$.

Do problem *"vNSA03 scheme G"* from the "vNSA von Neumann Stability Analysis Problems" series.

2.3 ExPD23 statement: Separation of variables for the Laplace equation in a circle (Neumann boundary conditions).

Consider the question of determining the steady state temperature in a thin circular plate such that: (a) The temperature flux is prescribed at the edges of the plate, and (b) The facets of the plate (top and bottom) plate are insulated. This problem can be written, using polar coordinates, in the form[2]

$$\frac{1}{r^2}\,(r\,(r\,T_r)_r + T_{\theta\theta}) = \Delta\, T = 0, \quad \text{for } \ 0 \leq \theta \leq 2\pi \quad \text{and } \ 0 \leq r \leq 1, \tag{2.5}$$

with the boundary condition[3]

$$T_r(1,\,\theta) = h(\theta), \tag{2.6}$$

where $\boldsymbol{h = h(\theta)}$ is some prescribed function such that

$$\int_0^{2\pi} h(\theta)\,d\theta = 0. \tag{2.7}$$

Equation (2.7) follows because unless the net heat flux through the boundary vanishes, no steady state temperature is possible. Equivalently: (2.5–2.6) does not have a solution unless (2.7) applies. Finally, we point out that the solution to (2.5–2.7) is defined only up to an arbitrary additive constant.[4] Thus, in order to obtain a unique solution, we impose the additional condition:

$$\text{The temperature } T \text{ vanishes at the origin.} \tag{2.8}$$

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[2]Here we use non-dimensional variables, selected so that the plate has radius 1.

[3]Of course, both T and h must be periodic of period 2π in θ.

[4]If the total heat flux through the boundary is zero, the total amount of heat in the plate is a constant, determined by the initial conditions.

The solution to (2.5–2.8) can be written as a (possibly infinite) sum of separation of variables solutions. **Separation of variables** solutions are product solutions of the form

$$T(r,\,\theta) = \phi(r)\,\psi(\theta), \tag{2.9}$$

where $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are some functions. In particular, from (2.6), it must be that

$$h(\theta) = \phi'(1)\,\psi(\theta), \tag{2.10}$$

where the prime denotes differentiation. However, for the separated solutions, the function $\boldsymbol{h}$ **is NOT prescribed:** It is whatever the form in (2.9) allows. The general h is obtained by adding separation of variables solutions.

These are the tasks in this exercise:

1. **Find ALL the separation of variables solutions.**
2. In exercise ExPD04 we stated that all the solutions to the Laplace equation — i.e.: (2.5) — must have the form $\boldsymbol{T = f(x - i\,y) + g(x + i\,y)}$, for some functions $\boldsymbol{f}$ and $\boldsymbol{g}$. *Note that these two functions must make sense for complex arguments, and have derivatives in terms of these complex arguments.* — which means that f and g must be analytic functions.

   Show that the result in the paragraph above applies to all the separated solutions found in item 1. Namely: find the functions f and g that correspond to each separated solution. *Note that in polar coordinates:* $\boldsymbol{z = x + i\,y = r\cos\theta + i\,r\sin\theta = r\,e^{i\theta}}$ *and* $\boldsymbol{\bar{z} = x - i\,y = r\cos\theta - i\,r\sin\theta = r\,e^{-i\theta}}$.

2.4 ExPD24 statement: A boundary value Problem for the heat equation.

Consider the following boundary value problem for the heat equation

$$T_t = T_{x\,x}, \quad \text{for} \quad 0 < x < \pi \;\text{ and }\; t > 0, \tag{2.11}$$

where

$$T(0,\,t) = 0 \quad \text{and} \quad T(\pi,\,t) = \cos(t) \quad \text{for} \quad t > 0. \tag{2.12}$$

T also satisfies some initial value, but we ignore this for now.

These are YOUR TASKS in this problem:

1. Use separation of variables to produce a solution to (2.11 – 2.12). Just **one** solution; any solution is O.K. [i.e., you do not have to worry about the initial data].

Hint #1. It will simplify things if you write $T = Re(u)$, where u satisfies the same problem as T does, but with the cosine replaced by $e^{i\,t}$. Then solve for the complex valued u.

Hint #2. Let $\boldsymbol{\nu} = \dfrac{\mathbf{1}+\boldsymbol{i}}{\sqrt{\mathbf{2}}}$. Then $\nu^2 = i$. You will need ν.

2. Consider now the problem that consists of (2.11 – 2.12), plus an initial condition

$$T(x,\,0) = T_0(x) \quad \text{for} \quad 0 < x < \pi. \tag{2.13}$$

This problem cannot be solved using normal modes alone, because the normal modes for the heat equation decay exponentially in time — thus they cannot match the boundary condition $T(\pi,\,t) = \cos(t)$. **Use the solution obtained in item 1 to reduce this problem to one that can be solved using normal modes — specifically, the modes $\left\{e^{-n^2\,t}\,\sin(n\,x)\right\}_{n=1}^{\infty}$.**

THE END.