18.311 — MIT (Spring 2012)

Rodolfo R. Rosales (MIT, Math. Dept., 2-337, Cambridge, MA 02139).

May 05, 2012.

Problem Set # 05.

Due: Friday May 11. Turn it in before 3:00 PM, in the box provided in Room 2-108.

IMPORTANT: The Regular and the Special Problems must be **stapled in TWO SEPARATE packages,** each with your **FULL NAME** clearly spelled.

Contents

1 Regular Problems.

Points: §1.1, 15 points. §1.2, 40 points. §1.3, 50 points. §1.4, 15 points.

1.1 ExPD04 statement: Complex variables solution of the 2D Laplace equation.

By direct substitution, show that

$$u = f(x - iy) + g(x + iy), \tag{1.1}$$

(where f and g are "arbitrary" functions) is a solution of Laplace's equation:

$$u_{xx} + u_{yy} = 0. \tag{1.2}$$

In fact, it can be shown that **all the solutions to the Laplace equation (1.2) have the form in (1.1)** — though you are not asked to do this here.

Note: The functions f and g must make sense for complex arguments, and have derivatives in terms of these complex arguments. Thus: **f and g must be analytic functions.**

1.2 ExPD22 statement: Separation of variables for the Laplace equation in a circle (Dirichlet boundary conditions).

Consider the question of determining the steady state temperature in a thin circular plate such that: (a) The temperature is prescribed at the edges of the plate, and (b) The facets of the plate (top and bottom) plate are insulated. This problem can be written, using polar coordinates, in the form[1]

$$\frac{1}{r^2}\,\left(r\,(r\,T_r)_r + T_{\theta\theta}\right) = \Delta\,T = 0, \quad \text{for } 0 \leq \theta \leq 2\pi \quad \text{and } 0 \leq r \leq 1, \tag{1.3}$$

[1] Here we use non-dimensional variables, selected so that the plate has radius 1.

with the boundary condition[2]

$$T(1,\,\theta) = h(\theta), \tag{1.4}$$

where $\boldsymbol{h = h(\theta)}$ is some prescribed function. *The solution to (1.3–1.4) can be written as a (possibly infinite) sum of separation of variables solutions.*

Separation of variables solutions to (1.3–1.4) are product solutions of the form

$$T(r,\,\theta) = \phi(r)\,\psi(\theta), \tag{1.5}$$

where $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are some functions. In particular, from (1.4), it must be that

$$h(\theta) = \phi(1)\,\psi(\theta). \tag{1.6}$$

However, for the separated solutions, the function $\boldsymbol{h}$ **is NOT prescribed:** It is whatever the form in (1.5) allows. The general h is obtained by adding separation of variables solutions.

These are the tasks in this exercise:

1. **Find ALL the separation of variables solutions.**
2. In exercise ExPD04 we stated that all the solutions to the Laplace equation — i.e.: (1.3) — must have the form $\boldsymbol{T = f(x - i\,y) + g(x + i\,y)}$, for some functions $\boldsymbol{f}$ and $\boldsymbol{g}$. *Note that these two functions must make sense for complex arguments, and have derivatives in terms of these complex arguments.* — which means that f and g must be analytic functions.

   Show that the result in the paragraph above applies to all the separated solutions found in item 1. Namely: find the functions f and g that correspond to each separated solution. *Notice that, in polar coordinates:* $\boldsymbol{z = x + i\,y = r\cos\theta + i\,r\sin\theta = r\,e^{i\theta}}$ *and* $\boldsymbol{\bar{z} = x - i\,y = r\cos\theta - i\,r\sin\theta = r\,e^{-i\theta}}$.

1.3 WaEq02 statement: String with both ends tied, and periodicity.

Consider an elastic (homogeneous) string under tension, tied at both ends, started from some initial configuration. To simplify the situation, assume that all the motion is restricted to happen in a plane. After a proper adimensionalization, the situation can be modeled by the mathematical problem below for the wave equation in 1-D — where $u = u(x,\,t)$ is the displacement from equilibrium of the string,[3]

$$u_{tt} - u_{xx} = 0 \quad \text{for} \quad 0 < x < \pi \quad \text{and} \quad t > 0, \tag{1.7}$$

[2] Of course, both T and h must be periodic of period 2π in θ.

[3] This formulation neglects dissipation in the string.

with *boundary conditions* $u(0,\, t) = u(\pi,\, t) = 0$, and *initial data*[4]

$$u(x,\, 0) = U(x) \quad \text{and} \quad u_t(x,\, 0) = V(x) = W'(x), \tag{1.8}$$

for some functions $U = U(x)$ and $V = V(x)$, where $W(x) = \int_0^x V(s)\, ds$.

We will further assume that[5]

$$\left.\begin{array}{l} \textbf{1. } U \textit{ is twice continuously differentiable in } 0 \le x \le \pi, \textit{ and } U(0) = U(\pi) = 0. \\ \textbf{2. } V \textit{ is continuously differentiable in } 0 \le x \le \pi, \textit{ and } V(0) = V(\pi) = 0. \end{array}\right\} \tag{1.9}$$

A. **Show that** the solution has the form

$$\boldsymbol{u(x,\, t) = f(t - x) - f(t + x),} \tag{1.10}$$

for some function $f = f(\zeta)$, which must be defined for any $\zeta > -\pi$.

Hint 1: As shown in the lectures (e.g., using the theory of characteristics), the general solution to the wave equation (1.7) has the form $u = \tilde{f}(x - t) + \tilde{g}(x + t)$, where $\tilde{f}$ and $\tilde{g}$ are "arbitrary" functions. Introduce another pair of arbitrary functions f and g, related to the older one by $f(s) = \tilde{f}(-s)$ and $g(s) = -\tilde{g}(s)$. It follows that you can also write

$$u = f(t - x) - g(t + x). \tag{1.11}$$

Now use the boundary condition at $x = 0$, and the fact that the solution must be defined for $0 < x < \pi$ and $t > 0$.

B. **Show that** f is a **periodic function** (what is the period?), and **write an explicit formula** for f in terms of U and W.

Hint 2: First, use the boundary condition at $x = \pi$ to show that f must be periodic. Then use the initial data to determine f. Notice that this step will give you an equation that gives f' with two different formulas, valid on two contiguous intervals. Be careful when you integrate the equation, so that you get formulas that are consistent across the interval ends.

C. **Show that** u is periodic in time. What is the period?

D. **Show that** the conditions in (1.9) guarantee that f is twice differentiable. Thus the formula in (1.10) can be substituted directly into (1.7) to show that u satisfies the equation.

However, note that *f'' is (generally) not continuous.* **Where can continuity fail for f''?**

Hint 3: compute f'' for the case in item E to get a feeling for what happens in general.

[4] We use primes to denote derivatives.

[5] These conditions admit a more general formulation, but here we stick to this for simplicity.

E. Compute the function f for the special case when $U(x) = x\,(\pi - x)$ and $V(x) = 0$.

Note that, even if both U and V are smooth (infinite derivatives) for $0 \leq x \leq \pi$, the **solution u** in (1.10) will (generally) **not be smooth.** Some lines in space-time (*lines of singularity*) will exist along which smoothness fails (some derivatives do not exist).

F. For the example in item **E, find the lines of singularity,** and **describe** the situation along them (e.g., which derivatives fail to exist).

Note: you will find that the lines of singularity are characteristics. This is generic, and not restricted to this particular example. In fact, this is an important property of the characteristics: they are the locus in space time where the solutions can be "singular". *This property allows the generalization of the notion of characteristic to situations where the definition "curves along which equations involving directional derivatives only can be written" is too restrictive (e.g., pde-s in more than 1-D).*

G. Finally, **what does the existence of these lines of singularity in the solution to the wave equation mean for the behavior of an actual string?** What are your thoughts on this?

1.4 WaEq03 statement: String with both ends tied equivalent to periodic and odd.

Consider an elastic (homogeneous) string under tension, tied at both ends, started from some initial configuration. To simplify the situation, assume that all the motion is restricted to happen in a plane. After a proper adimensionalization, the situation can be modeled by the mathematical problem below for the wave equation in 1-D — where $u = u(x,\,t)$ is the displacement from equilibrium of the string,[6]

$$u_{tt} - u_{xx} = 0 \quad \text{for} \quad 0 < x < \pi \quad \text{and} \quad t > 0, \tag{1.12}$$

with *boundary conditions* $u(0,\,t) = u(\pi,\,t) = 0$, and *initial data*

$$u(x,\,0) = U(x) \quad \text{and} \quad u_t(x,\,0) = V(x), \tag{1.13}$$

for some functions $U = U(x)$ and $V = V(x)$.

[6]This formulation neglects dissipation in the string.

In another problem (WaEq02) is is shown that:[7] The solution to (1.12 – 1.13) has the form

$$\left.\begin{array}{c} \boldsymbol{u(x,\,t) = f(t-x) - f(t+x)}, \\ \textbf{for some function } \boldsymbol{f = f(\zeta)}\textbf{, which is periodic of period } \boldsymbol{2\,\pi}. \end{array}\right\} \tag{1.14}$$

Vice-versa, it should be clear that any function of this form satisfies the problem in (1.12 – 1.13), with

$$U(x) = f(-x) - f(x) \quad \text{and} \quad V(x) = f'(-x) - f'(x), \tag{1.15}$$

where the primes are used to denote derivatives. Thus **(1.12 – 1.13) and (1.14) are equivalent.**

Finally, notice that u — as defined by (1.14) — satisfies:

$$\left.\begin{array}{l} \boldsymbol{u} \textbf{ is a solution to the wave equation in (1.12) which is both} \\ \textbf{periodic, of period } \boldsymbol{2\,\pi} \textbf{ in } \boldsymbol{x}\textbf{, as well as an odd function of } \boldsymbol{x}. \end{array}\right\} \tag{1.16}$$

Show that:

Let u satisfy (1.16). Then u satisfies a problem of the form (1.12 – 1.13). (1.17)

It then follows that **(1.16) and (1.12 – 1.13) are equivalent** — note the following chain of implications given by the above arguments: (1.12 – 1.13) $\Longleftrightarrow$ (1.14) $\Longrightarrow$ (1.16) $\Longrightarrow$ (1.12 – 1.13). You are being asked to prove the last one of these — i.e., (1.17).

---

2 Special Problems.

Points: §**2.1, 40 points.** §**2.2, 30 points.**

2.1 ExPD41 statement: Normal modes for the heat equation with Robin boundary conditions.

Consider the problem

$$T_t = T_{xx} \quad \text{for } \; 0 < x < \pi, \tag{2.18}$$

with boundary conditions[8] $T(0,\,t) = 0$ and $T_x(\pi,\,t) = T(\pi,\,t)$. Then

[7] Notice that, as far as providing a solution to (1.12 – 1.13), $f = f(\zeta)$ needs to be defined for $\zeta > -\pi$ only. However, because of the periodicity, we can assume that it is defined everywhere.

[8] Fixed temperature on the left. The right boundary condition is known as a Robin condition. It arises in situations where the heat flux at a boundary is proportional to the temperature difference with some cooling fluid flowing by.

1. **Find all the normal mode solutions** — that is: non-trivial solutions of the form $T = e^{\lambda t}\,\phi(x)$, where λ is a constant.

2. **Show that** the associated eigenvalue problem[9] is self-adjoint. Hence the spectrum is real, and the eigenfunctions can be used to represent "arbitrary" functions. In this case the spectrum is a discrete set of eigenvalues $\{\lambda_n\}$, and the eigenfunctions form an orthogonal basis.

3. Given initial conditions $T(x, 0) = T_0(x)$, **write the solution to (2.18) as a series** involving the normal modes
$$T = \sum_n a_n\, e^{\lambda_n t}\, \phi_n(x). \tag{2.19}$$
Write explicit expressions[10] **for the coefficients a_n.**

2.2 ExPD42 statement: String with both ends tied equivalent to periodic and odd.

Consider an elastic (homogeneous) string under tension, tied at both ends, started from some initial configuration. To simplify the situation, assume that all the motion is restricted to happen in a plane. After a proper adimensionalization, the situation can be modeled by the mathematical problem below for the wave equation in 1-D — where $u = u(x,\, t)$ is the displacement from equilibrium of the string,[11]

$$u_{tt} - u_{xx} = 0 \quad \text{for} \quad 0 < x < \pi \quad \text{and} \quad t > 0, \tag{2.20}$$

with *boundary conditions* $u(0,\, t) = u(\pi,\, t) = 0$, and *initial data*

$$u(x,\, 0) = U(x) \quad \text{and} \quad u_t(x,\, 0) = V(x), \tag{2.21}$$

for some functions $U = U(x)$ and $V = V(x)$.

Using a normal modes expansion[12] for the solution, **SHOW THAT:**

A. The solution to (2.20 – 2.21) can be written in the form

$$\left.\begin{array}{c} \boldsymbol{u(x,\, t) = f(t-x) - f(t+x),} \\ \textbf{for some function } \boldsymbol{f = f(\zeta)}\textbf{, which is periodic of period } \boldsymbol{2\,\pi}\textbf{.} \end{array}\right\} \tag{2.22}$$

[9] That is: $\lambda\,\phi = \phi_{xx}$, with the appropriate boundary conditions.
[10] They should be given by integrals involving T_0 and the eigenfunctions ϕ_n.
[11] This formulation neglects dissipation in the string.
[12] Do not worry about convergence when manipulating the various Fourier series that will arise in the process.

Exhibit the appropriate Fourier series that defines f.

Note: it should be clear that any function of this form satisfies the problem in (2.20 – 2.21), with

$$U(x) = f(-x) - f(x) \quad \text{and} \quad V(x) = f'(-x) - f'(x), \tag{2.23}$$

where the primes are used to denote derivatives. Thus (2.20 – 2.21) and (2.22) are equivalent.

B. The solution to (2.20 – 2.21) satisfies:

$$\textbf{As a function of } \boldsymbol{x}\textbf{, } \boldsymbol{u} \textbf{ is both } \boldsymbol{2\pi}\textbf{-periodic, as well as odd.} \tag{2.24}$$

Finally, **SHOW THAT:**

C. Let u be a solution to the wave equation in (2.20) that satisfies (2.24). Then u also satisfies the boundary conditions stated below (2.20).

Hints:

1. It is convenient to write the normal mode solutions in the form $u = e^{i\omega t}\,\phi(x)$. Then ϕ satisfies an eigenvalue problem in which $\lambda = -\omega^2$ is the eigenvalue. Thus for every eigenvalue you get two normal modes. This is crucial, since there are two initial data functions, U and V, that the solution must satisfy.

2. Recall that $\boldsymbol{e^{i\theta} = \cos\theta + i\,\sin\theta}$, $\boldsymbol{\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)}$, and $\boldsymbol{\sin\theta = \frac{1}{2\,i}\left(e^{i\theta} - e^{-i\theta}\right)}$. With these formulas you do not need to use/remember any trigonometric identities.

3. Any square integrable function, defined for $0 < x < \pi$, can be expanded in a sine-Fourier series:

$$F(x) = \sum_{n=1}^{\infty} s_n\,\sin(n\,x), \tag{2.25}$$

where $\boldsymbol{s_n = \frac{2}{\pi}\int_0^{\pi} F(x)\,\sin(n\,x)\,dx}$. Note that (2.25), in particular, defines F as a 2π-periodic, and odd, function for all values $-\infty < x < \infty$.

4. Any 2π-periodic, square integrable, function can be expanded in a complex Fourier series:

$$F(x) = \sum_{n=-\infty}^{\infty} c_n\,e^{i\,n\,x}, \tag{2.26}$$

where $\boldsymbol{c_n = \frac{1}{2\pi}\int_0^{2\pi} F(x)\,e^{-i\,n\,x}\,dx}$.

Supplementary Materiel.

3 Short notes on separation of variables.

Imagine that you are trying to solve a problem whose solution is a function of several variables, e.g.: $u = u(x,\, y\, z)$. Finding this solution directly may be very hard, but you may be able to find other solutions with the simpler structure

$$u = f(x)\, g(y)\, h(z), \tag{3.1}$$

where f, g, and h are some functions. This is called a *separated variables* solution.

If the problem that you are trying to solve is linear, and you can find enough solutions, of the form in (3.1), then you may be able to solve the problem using a linear combinations of separated variables solutions.

The technique described in the paragraph above is called *separation of variables*. Note that

1. When the problem is a pde, and if solutions of the form in (3.1), then the pde will reduce to three ode — one for each of f, g, and h. Thus the solution process is enormously simplified.

2. In (3.1) all the three variables are separated. But it is also possible to seek for *partially separated* solutions. For example

$$u = f(x)\, v(y,\, z). \tag{3.2}$$

This is what happens when you look for *normal mode* solutions to time evolution equations of the

$$u_t = \mathcal{L}\, u, \tag{3.3}$$

where $\mathcal{L}$ is some linear operator acting on the space variables $\vec{r} = (x, y, \dots)$ only — for example: $\mathcal{L} = \partial_x^2 + \partial_y^2 + \dots$. The normal mode solutions have time separated

$$u = e^{\lambda\, t}\phi(\vec{r}), \quad \text{where} \quad \lambda = \text{constant}, \tag{3.4}$$

and reduce the equation to an eigenvalue problem in space only

$$\lambda\phi = \mathcal{L}\, \phi. \tag{3.5}$$

3. *Separation of variables does not always work.* In fact, it rarely works if you just think of random problems. But it works for many problems of physical interest. For example: it works for the heat equation, but only for a few very symmetrical domains (rectangles, circles, cylinders, ellipses). But these are enough to build intuition as to how the equation works. Many properties valid for generic domains can be gleaned from the solutions in these domains.

4. Even if you cannot find enough separated solutions to write all the solutions as linear combinations of them, or if the problem is nonlinear and you can get just a few separated solutions, sometimes this is enough to discover interesting physical effects, or gain intuition as to the system behavior.

3.1 Example: heat equation in a square, with zero boundary conditions.

Consider the problem

$$T_t = \Delta\, T = T_{xx} + T_{yy}, \tag{3.6}$$

in the square domain $0 < x,\, y < \pi$, with T vanishing along the boundary, and with some initial data $T(x,\, y,\, 0) = W(x,\, y)$. To solve this problem by separation of variables, we first look for solutions of the form

$$T = f(t)\, g(x)\, h(y), \tag{3.7}$$

which satisfy the boundary conditions, but **not** *the initial data.* Why this? This is **important!:**

A. We would like to solve the problem for generic initial data, while solutions of the form (3.7) can only achieve initial data for which $W = f(0)\, g(x)\, h(y)$. This is very restrictive, even more so because (as we will soon see) the functions g and h will be very special. To get generic initial data, we have no hope unless we use arbitrary linear combinations of solutions like (3.7).

B. Arbitrary linear combinations of solutions like (3.7) will satisfy the boundary conditions if an only if each of them satisfies them.

Substituting (3.7) into (3.6) yields

$$f'\, g\, h = f\, g''\, h + f\, g\, h'', \tag{3.8}$$

where the primes indicate derivatives with respect to the respective variables. Dividing this through by u now yields

$$\frac{f'}{f} = \frac{g''}{g} + \frac{h''}{h}. \tag{3.9}$$

Since each of the terms in this last equation is a function of a different independent variable, the equation can be satisfied only if each term is a constant. Thus

$$\frac{g''}{g} = c_1, \quad \frac{h''}{h} = c_2, \quad \text{and} \quad \frac{f'}{f} = c_1 + c_2, \tag{3.10}$$

where c_1 and c_2 are constants. Now the problem has been reduced to a set of three simple ode. Furthermore, for (3.7) to satisfy the boundary conditions in (3.6), we need:

$$g(0) = g(\pi) = h(0) = h(\pi) = 0, \tag{3.11}$$

which restricts the possible choices for the constants c_1 and c_2. The equations in (3.10 – 3.11) are easily solved, and yield[13]

$$g = \sin(n\,x), \quad \text{with} \quad c_1 = -n^2, \tag{3.12}$$

$$h = \sin(m\,x), \quad \text{with} \quad c_2 = -m^2, \tag{3.13}$$

$$f = e^{-(n^2+m^2)\,t}, \tag{3.14}$$

where $n = 1,\, 2,\, 3,\, \ldots$ and $m = 1,\, 2,\, 3,\, \ldots$ are natural numbers. The solution to the problem in (3.6) can then be written in the form

$$T = \sum_{n,\,m=1}^{\infty} w_{nm}\, \sin(n\,x)\, \sin(n\,y)\, e^{-(n^2+m^2)\,t}, \tag{3.15}$$

where the coefficients w_{nm} follow from the double sine-Fourier series expansion (of the initial data)

$$W = \sum_{n,\,m=1}^{\infty} w_{nm}\, \sin(n\,x)\, \sin(n\,y). \tag{3.16}$$

That is

$$w_{nm} = \frac{4}{\pi^2} \int_0^{\pi} dx \int_0^{\pi} dy\, W(x,\, y)\, \sin(n\,x)\, \sin(n\,y). \tag{3.17}$$

[13] We set the arbitrary multiplicative constants in each of these solutions to one. Given (3.15 – 3.16), there is no loss of generality in this.

3.2 Example:
Heat equation in a circle, with zero boundary conditions.

We now, again, consider the heat equation with zero boundary conditions, but on a circle instead of a square. That is, using polar coordinates, we want to solve the problem

$$T_t = \Delta T = \frac{1}{r^2}\,(r(r\,T_r)_r + T_{\theta\theta})\,, \tag{3.18}$$

for $0 \le r < 1$, with $T(1,\,\theta,\,t) = 0$, and some initial data $T(r,\,\theta,\,0) = W(r,\,\theta)$. To solve the problem using separation of variables, we look for solutions of the form

$$T = f(t)\,g(r)\,h(\theta), \tag{3.19}$$

which satisfy the boundary conditions, but **not** *the initial data.* The reasons for this are the same as in items **A** and **B** above — see § 3.1. In addition, note that

C. We must use polar coordinates, otherwise solutions of the form (3.19) cannot satisfy the boundary conditions. This exemplifies an *important feature of the separation of variables method:* The separation must be done in a coordinate system where the boundaries are coordinate surfaces.

Substituting (3.19) into (3.18) yields

$$f'\,g\,h = \frac{1}{r^2}\,(f\,r\,(r\,g')'\,h + f\,g\,h'')\,, \tag{3.20}$$

where, as before, the primes indicate derivatives. Dividing this through by u yields

$$\frac{f'}{f} = \frac{(r\,g')'}{r\,g} + \frac{h''}{r^2\,h}. \tag{3.21}$$

Since the left side in this equation is a function of time only, while the right side is a function of space only, the two sides must be equal to the same constant. Thus

$$f' = -\lambda\,f \tag{3.22}$$

and

$$\frac{r\,(r\,g')'}{g} + \lambda\,r^2 + \frac{h''}{h} = 0, \tag{3.23}$$

where λ is a constant. Here, again, we have a situation involving two functions of different variables being equal. Hence

$$h'' = \mu\,h, \tag{3.24}$$

and

$$\frac{1}{r}\,(r\,g')' + \left(\lambda + \frac{\mu}{r^2}\right)\,g = 0, \tag{3.25}$$

where μ is another constant. The problem has now been reduced to a set of three ode. Furthermore, from the boundary conditions and the fact that θ is the polar angle, we need:

$$g(1) = 0 \qquad \text{and} \qquad h \;\text{ is periodic of period }\; 2\,\pi. \tag{3.26}$$

In addition, g must be non-singular at $r = 0$ — the singularity that appears for $r = 0$ in equation (3.25) is due to the coordinate system singularity, equation (3.18) is perfectly fine there.

It follows that it should be $\boldsymbol{\mu = -n^2}$ and

$$h = e^{i\,n\,\theta}, \quad \text{where } n \text{ is an integer.} \tag{3.27}$$

Notes:

- As in § 3.1, here and below, we set the arbitrary multiplicative constants in each of the ode solutions to one. As before, there is no loss of generality in this.
- Instead of complex exponentials, the solutions to (3.24) could be written in terms of sine and cosines. But complex exponentials provide a more compact notation.

Then (3.25) takes the form

$$\frac{1}{r}\,(r\,g')' + \left(\lambda - \frac{n^2}{r^2}\right)\,g = 0. \tag{3.28}$$

This is a Bessel equation of integer order. The non-singular (at $r = 0$) solutions of this equation are proportional to the Bessel function of the first kind $J_{|n|}$. Thus we can write

$$g = J_{|n|}(\kappa_{|n|\,m}\,r), \quad \text{and} \quad \lambda = \kappa^2_{|n|\,m}, \tag{3.29}$$

where $m = 1,\, 2,\, 3,\, \ldots$, and **$\boldsymbol{\kappa_{|n|\,m} > 0}$ is the $\boldsymbol{m}$-th zero of $\boldsymbol{J_{|n|}}$.**

Remark 3.1 *That (3.28) turns out to be a well known equation should not be a surprise. Bessel functions, and many other special functions, were first introduced in the context of problems like the one here — i.e., solving pde (such as the heat or Laplace equations) using separation of variables.*

Putting it all together, we see that the solution to the problem in (3.18) can be written in the form

$$T = \sum_{n=-\infty}^{\infty}\,\sum_{m=1}^{\infty} w_{n\,m}\,J_{|n|}(\kappa_{|n|\,m}\,r)\,\exp\left(i\,n\,\theta - \kappa^2_{|n|\,m}\,t\right), \tag{3.30}$$

where the coefficients $w_{n\,m}$ follow from the double (Complex Fourier) – (Fourier – Bessel) expansion

$$W = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} w_{n\,m}\, J_{|n|}(\kappa_{|n|\,m}\, r)\, e^{i\,n\,\theta}. \tag{3.31}$$

That is:

$$w_{n\,m} = \frac{1}{\pi\, J^2_{|n|+1}(\kappa_{|n|\,m})} \int_0^{2\,\pi} d\theta \int_0^1 r\, dr\, W(r,\,\theta)\, J_{|n|}(\kappa_{|n|\,m}\, r)\, e^{-i\,n\,\theta}. \tag{3.32}$$

Remark 3.2 *You may wonder how (3.32) arises. Here is a sketch:*

(i) For θ we use a Complex-Fourier series expansion. For any $2\,\pi$-periodic function

$$G(\theta) = \sum_{n=-\infty}^{\infty} a_n\, e^{i\,n\,\theta}, \quad \textit{where} \quad a_n = \frac{1}{2\,\pi} \int_0^{2\,\pi} G(\theta)\, e^{-i\,n\,\theta}\, d\theta. \tag{3.33}$$

(ii) For r we use the Fourier-Bessel series expansion explained in item (iii).

(iii) Note that (3.28), **for any fixed n,** *is an eigenvalue problem in $0 < r < 1$. Namely*

$$\mathcal{L}\, g = \lambda\, g, \quad \textit{where} \quad \mathcal{L}\, g = -\frac{1}{r}\,(r\, g')' + \frac{n^2}{r^2}\, g, \tag{3.34}$$

g is regular for $r = 0$, and $g(1) = 0$. Without loss of generality, assume that $n \geq 0$, and consider the set of all the (real valued) functions such that $\int_0^1 \tilde{g}^2(r)\, r\, dr < \infty$. Then define the scalar product

$$< \tilde{f},\, \tilde{g} >= \int_0^1 \tilde{f}(r)\, \tilde{g}(r)\, r\, dr. \tag{3.35}$$

With this scalar product $\mathcal{L}$ is self-adjoint, and it yields a complete set of orthonormal eigenfunctions

$$\phi_m = J_n(\kappa_{n\,m}\, r) \quad \textit{and} \quad \lambda_m = \kappa^2_{n\,m}, \quad \textit{where} \quad m = 1,\, 2,\, 3,\, \dots \tag{3.36}$$

Thus one can expand

$$F(r) = \sum_{m=1}^{\infty} b_m\, \phi_m(r), \quad \textit{where} \quad b_m = \frac{1}{\int_0^1 r\, \phi_m^2(r)\, dr} \int_0^1 F(r)\, \phi_m(r)\, r\, dr. \tag{3.37}$$

Finally, note that

$$\int_0^1 r\, \phi_m^2(r)\, dr \qquad \textit{follows from} \qquad \int_0^1 r\, J_n^2(\kappa_{n\,m}\, r)\, dr = \frac{1}{2}\, J^2_{n+1}(\kappa_{n\,m}). \tag{3.38}$$

We will not prove this identity here.

3.3 Example: Laplace equation in a circle sector, with Dirichlet boundary conditions, non-zero on one side.

Consider the problem

$$0 = \Delta u = \frac{1}{r^2}\left(r(r\,u_r)_r + u_{\theta\theta}\right), \quad 0 < r < 1 \quad \text{and} \quad 0 < \theta < \alpha, \tag{3.39}$$

where α **is a constant, with** $0 < \alpha < 2\pi$. The boundary conditions are

$$u(1,\,\theta) = 0, \quad u(r,\,0) = 0, \quad \text{and} \quad u(r,\,\alpha) = w(r), \tag{3.40}$$

for some given function w. To solve the problem using separation of variables, we look for solutions of the form

$$u = g(r)\,h(\theta), \tag{3.41}$$

which satisfy the boundary conditions at $r = 1$ *and* $\theta = 0$*, but* **not** *the boundary condition at* $\theta = \alpha$. The reasons are as before: we aim to obtain the solution for general w using linear combinations of solutions of the form (3.41) — see items **A** and **B** in § 3.1, as well as item **C** in § 3.2.

Substituting (3.41) into (3.39) yields, after a bit of manipulation

$$\frac{r\,(r\,g')'}{g} + \frac{h''}{h} = 0. \tag{3.42}$$

Using the same argument as in the prior examples, we conclude that

$$r\,(r\,g')' - \mu\,g = 0 \quad \text{and} \quad h'' + \mu\,h = 0, \tag{3.43}$$

where μ is some constant, $g(1) = 0$, and $h(0) = 0$. Again: the problem is reduced to solving ode. In fact, it is easy to see that it should be[14]

$$h = \frac{1}{s}\sinh(s\,\theta) \quad \text{and} \quad g = \frac{r^{i\,s} - r^{-is}}{s}, \quad \text{where} \quad \mu = -s^2, \tag{3.44}$$

and as yet we know nothing about s, other than it is some (possibly complex) constant.

In § 3.2, we argued that g *should be non-singular at* $r = 0$*,* since the origin was no different from any other point inside the circle for the problem in (3.18) — the singularity in the equation for g in (3.25) is merely a consequence of the coordinate system singularity at $r = 0$. **We cannot make**

[14] The multiplicative constant in these solutions is selected so that the correct solution is obtained for $s = 0$.

this argument here, since the origin is on the boundary for the problem in (3.39 – 3.40) — *there is no reason why the solutions should be differentiable across the boundary!* We can only state that

$$\boldsymbol{g} \textbf{ should be bounded} \quad \Longleftrightarrow \quad \boldsymbol{s \neq 0} \textbf{ is real.} \tag{3.45}$$

Furthermore, exchanging s by $-s$ does not change the answer in (3.44). Thus

$$\boldsymbol{0 < s < \infty.} \tag{3.46}$$

In this example we end up with a continuum of separated solutions, as opposed to the two prior examples, where discrete sets occurred.

Putting it all together, we now write the solution to the problem in (3.39 – 3.40) as follows

$$u = \int_0^\infty \frac{\sinh(s\,\theta)}{\sinh(s\,\alpha)} \left(r^{i\,s} - r^{-i\,s}\right) W(s)\,ds, \tag{3.47}$$

where W is computed in (3.50) below.

Remark 3.3 *Start with the complex Fourier Transform*

$$f(\zeta) = \int_{-\infty}^\infty \hat{f}(s)\,e^{i\,s\,\zeta}\,ds, \quad \textit{where} \quad \hat{f}(s) = \frac{1}{2\,\pi}\int_{-\infty}^\infty f(\zeta)\,e^{-i\,s\,\zeta}\,d\zeta. \tag{3.48}$$

Apply it to an odd function. The answer can then be manipulated into the sine Fourier Transform

$$f(\zeta) = \int_0^\infty F(s)\,\sin(s\,\zeta)\,ds, \quad \textit{where} \quad F(s) = \frac{2}{\pi}\int_0^\infty f(\zeta)\,\sin(s\,\zeta)\,d\zeta, \tag{3.49}$$

and $0 < \zeta,\ s < \infty$. *Change variables, so that* $\boldsymbol{0 < r = e^{-\zeta} < 1.}$ *Then, with* $w(r) = f(\zeta)$ *and* $W(s) = -\dfrac{1}{2\,i}\,F(s)$, *this yields*

$$\begin{aligned} \boldsymbol{w(r)} &= \int_0^\infty \boldsymbol{W(s)\,\left(r^{i\,s} - r^{-i\,s}\right)\,ds}, \quad \textit{where} \\ \boldsymbol{W(s)} &= \frac{\boldsymbol{1}}{\boldsymbol{2\,\pi}}\int_0^1 \boldsymbol{\left(\frac{r^{-i\,s} - r^{i\,s}}{r}\right)\,w(r)\,dr}, \end{aligned} \tag{3.50}$$

which is another example of a transform pair associated with the spectrum of an operator *(see below).*

Finally: What is behind (3.50)? Why should we expect something like this? Note that the problem for g can be written in the form

$$\mathcal{L}\,g = \mu\,g, \quad \text{where} \quad \mathcal{L}\,g = r\,(r\,g')', \tag{3.51}$$

$g(1) = 0$ and g is bounded (more accurately: the inequality in (3.52) applies). This is an eigenvalue problem in $0 < r < 1$. Further, consider consider the set of all the functions such that

$$\int_0^1 |\tilde{g}|^2(r)\,\frac{dr}{r} < \infty, \tag{3.52}$$

and define the scalar product

$$< \tilde{f},\, \tilde{g} >= \int_0^1 \tilde{f}^*(r)\,\tilde{g}(r)\,\frac{dr}{r}. \tag{3.53}$$

With this scalar product $\mathcal{L}$ is self-adjoint. However, it does not have any discrete spectrum,[15] only continuum spectrum — with the pseudo-eigenfunctions given in (3.44), for $0 < s < \infty$. This continuum spectrum is what is associated with the formulas in (3.50). In particular, note the presence of the scalar product (3.53) in the formula for W.

THE END.

[15] No solutions that satisfy $g(1) = 0$ and (3.52).