18.311 — MIT (Spring 2012)

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Problem Set # 04.

Due: Friday May 4. Turn it in before 3:00 PM, in the box provided in Room 2-108.

IMPORTANT: The Regular and the Special Problems must be **stapled in TWO SEPARATE packages,** each with your **FULL NAME** clearly spelled.

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1 Regular Problems.

Points: §1.1, 30 points. §1.2, 30 points. §1.3, 50 points. §1.4, 30 points. §1.5, 10 points.

1.1 Statement: Longitudinal vibrations of an elastic rod.

This problem consists of **5 questions that you must answer** (see below).

Consider a slender cylindrical rod of same elastic material, lined up with the x axis — i.e.: the axis of the cylinder lies along the x coordinate axis. Assume now that all the motion occurs along the x axis.[1] In particular, assume that there is neither bending nor torsion of the rod. In this situation, *we can describe the rod, at any any time, by a single function*

$$X = X(x,\, t) = \begin{cases} \text{Position on the } x \text{ axis, at time } t\text{, of the point in the rod} \\ \text{whose position is } x \text{ when the rod is at equilibrium.} \end{cases} \tag{1.1}$$

Here by equilibrium we mean that there are no internal (elastic) forces in the rod. Hence we can write $\boldsymbol{X = x + u(x,\, t)}$, where $\boldsymbol{u}$ denotes the displacement of any point in the rod from its equillibrium position. When the rod is not at equilibrium, forces occur. In the case of an elastic body, these forces depend on how stretched or compressed the rod is at each point. Before we can write an expression for the elastic forces, we **need a measure of the rod stretching:**

Consider what happens with an infinitesimal length dx, under the transformation $x \to X$. Clearly: $dx = (x + dx) - x \quad \longrightarrow \quad X(x + dx,\, t) - X(x,\, t) = (1 + u_x)\, dx$. Thus, at every point, $\boldsymbol{u_x}$ measures how stretched (or compressed, for $\boldsymbol{u_x < 0}$) the material of the rod is, with $\boldsymbol{1 + u_x}$ the ratio of the current length relative to the equilibrium length. *The (elastic) forces on the rod are then as follows:*

$$\left.\begin{array}{l} \text{At each point on the rod (say, the one that at equilibrium is at } x = x_*\text{), the portion} \\ \text{of the rod ahead (i.e.: } x > x_*\text{) applies a force } \boldsymbol{F} \text{ on the portion behind (i.e.: } x < x_*\text{).} \\ \text{Viceversa, } x < x_* \text{ applies an opposite and equal force } -\boldsymbol{F} \text{ on } x > x_*. \end{array}\right\} \tag{1.2}$$

QUESTION 1. Guive a physical argument for why the two forces must be equal and opposite. *Hint:* what would happen with the infinitesimal amount of mass at x_* if they were not?

Because the rod is elastic, F is a function of u_x, namely

$$F = F(u_x), \tag{1.3}$$

where we assume that the rod is homogeneous, so that the relationship between force and deformation is the same all along the rod.

Note 1. If the rod is stretched ($u_x > 0$) each side should pull on the other ($F > 0$). Viceversa, a push ($F < 0$) should result from compression ($u_x < 0$). In general, we expect that:

$$F(0) = 0, \qquad \text{and} \qquad \frac{dF}{du_x} > 0. \tag{1.4}$$

[1] This is an *approximation.* If a rod is stretched, it does (generally) become thinner. Thus motion perpendicular to the rod axis occurs. However, if the stretching is not too large, it is reasonable to neglect the transversal motion.

In particular, for small deformations (u_x small), we can approximate F by the first (significant) term in its Taylor expansion (assuming that F is smooth), and write

$$F \cong k\, u_x, \quad \text{where} \quad k > 0 \quad \text{is a constant.} \tag{1.5}$$

This is **Hooke's law of elasticity.**

Note 2. A uniform displacement of a rod at elastic equilibrium should keep the rod at equilibrium. This is borne out by the equations above, since a uniform displacement corresponds to $u = u(t)$. There are very many possible positions where a rod is at elastic equilibrium. In defining X above, we must select one of them. Of course, it is convenient[2] to select one in an inertial frame of reference, so that no 'inertial" forces appear when writing the equations for the rod behavior. Assume that this is the case here.

QUESTION 2. Using the methods introduced in class, derive an equation (a p.d.e.) for the function $u = u(x,\, t)$, assuming that the only forces affecting the rod behavior are the elastic forces described above. Assume that the mass per unit length of the rod at equilibrium is a constant, ρ.

Hints. Note that the velocity of any material point on the rod, characterized by its equilibrium position x, is given by $u_t(x,\, t)$. Thus the momentum in any piece of the rod, $a < x < b$, is given by $\int_a^b \rho\, u_t\, dx$. Write now an equation for the conservation of the momentum in $a < x < b$, then use the fact that $a < b$ are arbitrary to derive a p.d.e. for u.

QUESTION 3. What form does the p.d.e. you just derived have when (1.5) applies?

QUESTION 4. What dimensions does the constant $c = \sqrt{k/\rho}$ have?

QUESTION 5. In general, what dimensions does $c = c(u_x) = \sqrt{\dfrac{1}{\rho}\dfrac{dF}{du_x}}$ have? Notice that u_x is dimension-less.

Remark. An elastic rod can exhibit other types of motion, in addition to the longitudinal one described here. In particular: bending, and torsion (we may come back to these later in the course). Each of these is associated with a particular type of waves that can propagate along a rod. For example: P-waves (longitudinal vibrations), and S-waves (transverse vibrations).

[2] Though not required!

1.2 SGDP02 statement: Simple initial value problem.

Solve the equations[3] in (3.1), with $p = \dfrac{1}{2}\,\rho^2$, for the initial conditions

$$u \equiv -\frac{1}{3} \quad \text{and} \quad \rho \equiv \frac{1}{9} \quad \text{for} \quad x < 0, \tag{1.6}$$

$$u \equiv \frac{1}{3} \quad \text{and} \quad \rho \equiv \frac{4}{9} \quad \text{for} \quad x > 0. \tag{1.7}$$

Display explicit formulas, for u and ρ as functions of $(x,\, t)$, for all $-\infty < x < \infty$ and $t > 0$.

Hint: check the Riemann invariants — see equations (3.9 – 3.11).

Remark 1.1 *Notice that the form of the pressure here corresponds to a polytropic gas with $\gamma = 2$. Alternatively, the equations correspond to the shallow water system.*

1.3 WaEq01 statement:
Initial value problem and propagating singularities.

Consider an elastic (homogeneous) string under tension, tied at one end, initially at rest, and forced by a (small amplitude) harmonic shaking of the other end. To simplify the situation, assume that all the motion is restricted to happen in a plane.

After a proper adimensionalization, the situation is modeled by the mathematical problem below for the wave equation in 1-D — where $u = u(x,\, t)$ is the displacement from equilibrium of the string,[4]

$$u_{tt} - u_{xx} = 0, \quad \text{for} \quad 0 < x < 1, \quad \text{and} \quad t > 0, \tag{1.8}$$

with *initial data* $u(x,\, 0) = u_t(x,\, 0) = 0$, and *boundary conditions*

$$u(0,\, t) = 1 - \cos(\omega\, t) = \sigma(t) \quad \text{and} \quad u(1,\, t) = 0, \tag{1.9}$$

where ω is some constant, and σ is defined by the equation.

- **FIND** the solution to this problem, for the time interval $0 < t \leq 4$.

  Hint: Any solution to equation (1.8) must have the form $u = f(x - t) + g(c + t)$, for some functions f and g. Determine these functions, as needed to find the solution for $0 < t \leq 4$.

  Note that **the solution** that you will obtain in this fashion **is not infinitely differentiable** (although it has just enough derivatives to make sense as a solution). There are some lines in space-time (*lines of singularity*) along which smoothness fails (some derivatives do not exist).

[3] Note that here we use a non-dimensional formulation.

[4] This formulation neglects dissipation in the string.

- **FIND** the lines of singularity, and describe what the situation is along them: how it is that the solution is not smooth there, which derivatives fail to exist, etc.

  Note: you will find that the lines are characteristics. In fact, this is a very important property of characteristics: they are the locus in space time where the solutions can be singular. This property allows the generalization of characteristics to situations where the definition "curves along which equations involving directional derivatives only can be written" is too restrictive.

- **FINALLY:** what is special about the cases $\omega = \pi$ and $\omega = \pi/2$?

1.4 ExPD40 statement: Normal modes for the heat equation with mixed boundary conditions.

Consider the problem

$$T_t = T_{xx} \quad \text{for } \ 0 < x < \pi, \tag{1.10}$$

with boundary conditions[5] $T(0,\, t) = 0$ and $T_x(\pi,\, t) = 0$. Then

1. **Find all the normal mode solutions** — that is: non-trivial solutions of the form $T = e^{\lambda\, t}\, \phi(x)$, where λ is a constant.
2. **Show that** the associated eigenvalue problem[6] is self-adjoint. Hence the spectrum is real, and the eigenfunctions can be used to represent "arbitrary" functions. In this case the spectrum is a discrete set of eigenvalues $\{\lambda_n\}$, and the eigenfunctions form an orthogonal basis.
3. Given initial conditions $T(x, 0) = T_0(x)$, **write the solution to (1.10) as a series** involving the normal modes
$$T = \sum_n a_n\, e^{\lambda_n\, t}\, \phi_n(x). \tag{1.11}$$
   Write explicit expressions[7] for the coefficients a_n.

[5] Fixed temperature on the left and no heat flux on the right.
[6] That is: $\lambda\, \phi = \phi_{xx}$, with the appropriate boundary conditions.
[7] They should be given by integrals involving T_0 and the eigenfunctions ϕ_n.

1.5 ExPD50 statement: Normal modes do not always work (example 01).

Normal mode expansions are not guaranteed to work when the associated eigenvalue problem is not self adjoint (more generally, not normal[8]). Here is an example: consider the problem

$$u_t + u_x = 0 \quad \text{for } \ 0 < x < 1, \quad \text{with boundary condition } u(0,\, t) = 0. \tag{1.12}$$

Find all the normal mode solutions to this problem,[9] and **show that the general solution to (1.12) cannot be written as a linear combination of these modes.**

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2 Special Problems.

Each of the six parts in problem SGDP01 below is worth 10 points.

2.1 SGDP01 statement: The equations in Lagrangian coordinates.

As mentioned in § 3, in the form given by (3.1), the equations are said to be written in **Eulerian (or laboratory) coordinates.** The equations can also be written in **Lagrangian (or particle following) coordinates**, where the "space" coordinate is a label for each gas particle, rather than a position in space. The "label" used is, in fact, the *mass to the left* of each particle, as defined below.

To transform the equations in (3.1) to Lagrangian coordinates, first select some arbitrary (but fixed) fluid particle P, and let $x = x_p(t)$ be the position of the particle P at any time. Thus

$$\frac{dx_p}{dt} = u(x_p,\, t). \tag{2.1}$$

Next, introduce the change of variables

$$(x,\, t) \quad \longrightarrow \quad (\sigma,\, t) \tag{2.2}$$

where $\sigma = \sigma(x, t)$ is defined by

$$\sigma = \sigma(x, t) = \int_{x_p}^{x} \rho(\zeta,\, t)\, d\zeta. \tag{2.3}$$

Note that

[8]An operator A is normal if it commutes with its adjoint: $A\,A^\dagger = A^\dagger\,A$.

[9]That is, non-trivial solutions of the form $u = e^{\lambda\, t}\,\phi(x)$, where λ is a constant.

a. The variable $\boldsymbol{\sigma}$ has dimensions of mass. In fact: **$\boldsymbol{\sigma}$ is the amount of mass in the gas between $\boldsymbol{x_p}$ and $\boldsymbol{x}$, with $\boldsymbol{\sigma \geq 0}$ if $\boldsymbol{x \geq x_p}$, and $\boldsymbol{\sigma \leq 0}$ if $\boldsymbol{x \leq x_p}$.**

b. If there is a finite amount of gas, we can take $x_p \equiv -\infty$. In this case σ is the mass of the gas to the left of the point x, and it is always non-negative.

c. If there is an impermeable wall somewhere (beginning or end of a closed pipe containing the gas), we can use as x_p the position of this wall.

$$\left.\begin{array}{l}\textbf{For simplicity, in what follows we assume that } \rho > 0 \\ \textbf{everywhere. That is, that there is no vacuum anywhere.}\end{array}\right\} \tag{2.4}$$

Your tasks in this problem are stated in items 1 – 6 below. (2.5)

1. Show that: If $x = X(t)$ is a curve defined by $\sigma(X,\, t) = \text{constant}$, then $x = X(t)$ is a particle path. That is

$$\frac{dX}{dt} = u(X,\, t). \tag{2.6}$$

Vice-versa, if $x = X(t)$ is a particle path, then $\sigma(X,\, t) = \text{constant}$.

This shows that **$\boldsymbol{\sigma}$ is a Lagrangian Coordinate.** That is: it is constant when following a fixed mass point in the gas.

Hint: plug $x = X(t)$ into equation (2.3), and take the derivative with respect to time. Then use conservation of mass [the first equation in (3.1)] and (2.1).

2. Show that:

$$\frac{\partial\,\sigma}{\partial\,x} = \rho, \quad \text{and} \quad \frac{\partial\,\sigma}{\partial\,t} = -\rho\,u. \tag{2.7}$$

Hint: The equation on the left here is trivial. For the other equation use conservation of mass [the first equation in (3.1)] and (2.1).

3. Show that: For any given time, the transformation $x \to \sigma$ is invertible. Give a formula for the inverse, $\boldsymbol{x = \chi(\sigma,\, t)}$, assuming that you know the specific volume $\mathrm{v} = \dfrac{1}{\rho}$ in the Lagrangian coordinates, that is $\mathrm{v} = \mathrm{v}(\sigma,\, t)$.

Hint: Using the result from item 2, show that σ is a strictly increasing function of x — hence it has an inverse. Then compute χ_σ. You will get a formula that you can integrate to obtain χ. The result has the same "flavor" as (2.3) — that is, an integral involving v.

4. Show that: under the change of variables in (2.2), the (isentropic) Euler equations of Gas Dynamics in (3.1) take the form

$$\left.\begin{aligned} \mathrm{v}_t - u_\sigma &= 0, \\ u_t + p_\sigma &= 0, \end{aligned}\right\} \tag{2.8}$$

where $\mathbf{v} = \dfrac{\mathbf{1}}{\boldsymbol{\rho}}$ is the specific volume, and the pressure is a function[10] of $\mathbf{v}$ via (3.2).

Note that $\dfrac{\boldsymbol{dp}}{\boldsymbol{d}\mathbf{v}} = \boldsymbol{-\rho^2\, c^2}$, where c is the sound speed (defined in (3.3)).

The system of equations given by (2.8) is known as the

Conservation Form of the Isentropic Equations of Gas Dynamics, in Lagrangian Coordinates.

Hint: First, use the equations in (2.7) to obtain expressions, for the partial derivatives of any function f in the Eulerian frame, in terms of partial derivatives in the Lagrangian fame. Then use these expressions to re-write the equations in the Lagrangian frame. Note that the computations are a little simpler if the form of the equations in (3.4) is used — where, in addition, the substitution $c^2\,\rho_x = p_x$ is made in the second equation.

Remark 2.1 *Notice that the derivation of the system of equations in (2.8), as sketched by the hint above, involves manipulations with derivatives that are not valid when the solution is not smooth (i.e., shocks are present). Nevertheless, that the system in conservation form in (2.8) is valid, even when shocks are present, is easily checked:*

First: $\displaystyle\int_{\sigma_1}^{\sigma_2} \mathrm{v}\, d\sigma$ is the distance between the particles with Lagrangian coordinates σ_1 and σ_2. Hence the integral of the first equation in (2.8) states that this distance evolves according to the difference in velocity between these two gas particles. It follows that *the first equation in (2.8) expresses the "conservation of distance", and it is valid even for solutions that are not smooth.*

Second: $\displaystyle\int_{\sigma_1}^{\sigma_2} u\, d\sigma$ is the total momentum of the gas between the particles with Lagrangian coordinates σ_1 and σ_2. As above, it follows that *the second equation in (2.8) expresses the "conservation of momentum", and it is valid even for solutions that are not smooth.*

[10] That is: $p = p(\mathrm{v})$. For an ideal gas $p = \kappa\, \mathrm{v}^{-\gamma}$.

5. Using the first equation in (2.8) [*"conservation of distance"*], and the formula for $\chi = \chi(\sigma,\, t)$ that you derived in item 3, **obtain expressions for χ_σ and χ_t in terms of v and u.** These expressions are the analog, in Lagrangian coordinates, of (2.7) in Eulerian coordinates.

Hint: Note that x_p is the position (in Eulerian coordinates) of the particle with $\sigma = 0$ — hence $\dot{x_p}$ is equal to the value of the velocity u at $\sigma = 0$ (in Lagrangian coordinates).

6. Write the system in (2.8) in characteristic form — that is, the analog of (3.5 – 3.11).

Hint: Use that $\dfrac{dp}{d\mathrm{v}} = -\rho^2\, c^2$ to write (2.8) as a system of equations in v *and u. Then consider combinations of the two equations of the form* (second equation) + α (first equation)*, and select α to obtain the characteristic equations (all the derivatives are along a single direction in space-time).*

Supplementary Materiel.

3 The isentropic Euler equations of Gas Dynamics in 1-D.

Equations that govern the behavior of a gas can be derived using the same *Conservation Equation* techniques used to derive equations for the examples of Traffic Flow, River Flows, Shallow Water Waves, Modulations of Dispersive Waves, etc. The conserved quantities in this case are the mass, the momentum and the energy. The resulting equations are the **Euler equations of Gas Dynamics.**

Under certain conditions, one can assume that the entropy is a constant throughout the flow. Then the equations can be simplified, with the elimination of the equation for the conservation of energy. The **one dimensional isentropic (constant entropy) Euler equations of Gas Dynamics** are

$$\left.\begin{aligned} \rho_t + (\rho\, u)_x &= 0, \\ (\rho\, u)_t + (\rho\, u^2 + p)_x &= 0, \end{aligned}\right\} \tag{3.1}$$

where $\rho = \rho(x,\, t)$, $u = u(x,\, t)$, and $p = p(x,\, t)$ are the gas mass density, flow velocity, and pressure, respectively. The first equation here implements the conservation of mass, and the second the conservation of momentum. We also need an equation relating the fluxes of the conserved quantities with the conserved densities — the analogue of the equation $q = Q(\rho)$ in Traffic Flow. In this case this is provided by an **equation of state**, relating the pressure to the density. This takes the form

$$p = P(\rho), \quad \text{where } P \text{ is a function satisfying } \frac{dP}{d\rho} > 0. \tag{3.2}$$

For example, for an ideal gas $P = \kappa\rho^\gamma$, where $\kappa > 0$ and $1 < \gamma < 2$ are constants.

The system of equations given by (3.1) is known as the

Conservation Form of the Isentropic Equations of Gas Dynamics, in Eulerian Coordinates.

Remark 3.1 *Normally, the pressure is a function of both the density and some other thermodynamic variable, such as the temperature. But the isentropic assumption allows us to write the pressure as a function of the gas mass density only.*

Introduce now the function $c = c(x,\, t) > 0$ (**c is the sound speed)** by[11]

$$c = C(\rho), \quad \text{where } C(\rho) = \sqrt{\frac{dP}{d\rho}(\rho)}. \tag{3.3}$$

An alternative form of the equations in (3.1) is then given by

$$\left.\begin{aligned} \rho_t + u\,\rho_x + \rho\,u_x &= 0, \\ u_t + \frac{c^2}{\rho}\,\rho_x + u\,u_x &= 0. \end{aligned}\right\} \tag{3.4}$$

Multiplying the first equation here by c/ρ, and adding (or subtracting) it to the second, yields the **characteristic form** of the equations

$$0 = (u_t + (u+c)u_x) + \frac{c}{\rho}\left(\rho_t + (u+c)\rho_x\right), \tag{3.5}$$

$$0 = (u_t + (u-c)u_x) - \frac{c}{\rho}\left(\rho_t + (u-c)\rho_x\right). \tag{3.6}$$

Equivalently

$$0 = \frac{du}{dt} + \frac{c}{\rho}\frac{d\rho}{dt} \qquad \text{along} \qquad \frac{dx}{dt} = u + c, \tag{3.7}$$

$$0 = \frac{du}{dt} - \frac{c}{\rho}\frac{d\rho}{dt} \qquad \text{along} \qquad \frac{dx}{dt} = u - c. \tag{3.8}$$

[11]For an ideal gas, $c = \sqrt{\gamma\,p/\rho}$. For dry air at one atmosphere and 15 degrees Celsius: $p = 1.013 \times 10^6$ dyn/cm^2, $\rho = 1.226 \times 10^{-3}$ g/cm^3, and $\gamma = 1.401$. Hence $c = 340.2$ m/s — measured value is $c = 340.6$ m/s.

These last two equations show that, if we introduce a new variable $\boldsymbol{h = h(\rho)}$, defined by

$$\frac{dh}{d\rho} = \frac{C(\rho)}{\rho}, \tag{3.9}$$

then

$$u + h \quad \text{is constant along the characteristics} \quad \frac{dx}{dt} = u + c, \tag{3.10}$$

$$u - h \quad \text{is constant along the characteristics} \quad \frac{dx}{dt} = u - c. \tag{3.11}$$

The variable $\mathcal{R} = u + h$ is called the **right Riemann invariant**, while $\mathcal{L} = u - h$ is called the **left Riemann invariant.**

THE END.