18.311 — MIT (Spring 2012)
Problem Set # 03

Due Friday, April 13.

Turn it in before 3:00pm, in the box provided in Room 2-108
IMPORTANT: The Regular and the Special Problems must be stapled in **TWO SEPARATE packages**, each with your **FULL NAME** clearly spelled.

1 Regular problems:

1.1 Statement: Inviscid Burger's equation.

Solve the following initial value problems for the inviscid Burger's equation

$$u_t + uu_x = 0.$$

Draw the characteristic diagram and sketch the solutions at different times (initial time, before shocks, after shocks form). If there are shocks, find their speeds and check that they satisfy the Entropy condition.

1. $u(x, 0) = 1$ for $x < 0$,
   $u(x, 0) = 1 - x$ for $0 < x < 1$
   $u(x, 0) = 0$ for $x > 1$

2. $u(x, 0) = 0$ for $x < -1$,
   $u(x, 0) = -1$ for $-1 < x < 0$
   $u(x, 0) = 1$ for $0 < x < 1$
   $u(x, 0) = 0.6$ for $x > 1$.

3. $u(x, 0) = 2$ for $x < -4$
   $u(x, 0) = 1$ for $-4 < x < 0$
   $u(x, 0) = 1 - x$ for $0 < x < 1$
   $u(x, 0) = 0$ for $x > 1$.

1.2 Statement: First order PDEs (ref: lecture 11)

Solve the following problems using the method of characteristics. You would find a surface which satisfies the given equation, and passes through the given curve.
Note: In most of the problems the solution will not be an explicit function $u(x, y)$.

1. $y^2 u_x + xyu_y = x$, given by $x = 0$ and $u = y^2$

2. $xu_x - 2yu_y = x^2 + y^2$, given by $y = 1$, $u = x^2$

3. $xu_x + yu_y = 2xy$, given $y = x$, $u = x^2$

1.3 Statement: Laplace's method

1. Compute $\int_{-\infty}^{\infty} e^{-x^2} dx$

   Find the leading order behavior for these integrals

2. $\int_{0.5}^{3.5} \tanh(x) \exp(t(1 + \sin x))dx$ for large time ($t \to \infty$)
3. $\int_{-20}^{42} (x^2 + \sin x) \exp\left(-\frac{(x-3)^2}{4\nu t}\right) dx$ for vanishing viscosity ($\nu \to 0$)

2 Special problems

1. Find a surface that goes through the straight line $y = x$, $u = 1$, which is orthogonal to the surfaces $x^2 + y^2 + z^2 = Cx$ (C is an arbitrary constant).

   Hint: think about a PDE $au_x + bu_y = c$ as a scalar product of two vectors, one of which is a normal to the solution surface $u = u(x, y)$.

2. Explain which problems (a-d) could have wave-breaking for some initial conditions and some values of V or a? If so, at what time if the initial conditions are $\rho(x, 0) = F(x)$?

   (a) $\rho_t + V\rho_x = 0$, $V = const.$

   (b) $\rho_t + (a + \rho)\rho_x = 0$, $a = const.$

   (c) $\rho_t + \rho\rho_x = 4$.

   (d) $\rho_t + (1 + \rho)\rho_x = -\gamma\rho$, $\gamma = const > 0$.