18.311 — MIT (Spring 2012)
Problem Set # 02

March 16, 2012

Due: Friday, March 23.

Turn it in before 3:00pm, in the box provided in Room 2-108
IMPORTANT: The Regular and the Special Problems must be stapled in **TWO SEPARATE packages**, each with your **FULL NAME** clearly spelled.

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1 Regular problems:

1.1 Statement: Traffic flow, method of characteristics, shocks, and rarefections

Given the conservation law for cars

$$\rho_t + q_x = 0,$$

where q is the flux of cars, and ρ is the density of cars, and a particular form of the flux

$$q = u_{max}\rho(1 - \rho/\rho_{max}).$$

Assignment: Find the solution to the traffic flow problems, for which the initial conditions are stated below. The steps in solution that must be displayed are:

- Find the equation on characteristic and characteristic speeds
- Find the parametric form of the solution
- Plot the characteristics
- If there are shocks:
  - find the first time the shock forms
  - what is the speed of the shock?
  - is the Lax entropy condition satisfied?
  - plot the car density at the initial time, before the shock forms, and after the shock forms.
- Discuss if the problem physically in therms of car dynamics: traffic jam, green light, red light, heavy traffic (characteristic speed < 0), light traffic (char speed > 0), etc.

Traffic flow problems:

1. State the general facts
   (a) What is the formula for characteristic speed? What is characteristic speed graphically?
   (b) If there is a shock with ρ_1 behind the shock and ρ_2 ahead of the shock, what is the speed of propagation of the shock for the traffic flow with the given form of the flux?
   (c) State the entropy condition for this problem.
2. $\rho(x, 0) = \rho_{max}/2$ for $x < -L$,
   $\rho(x, 0) = \rho_{max}$ for $-L < x < 0$,
   $\rho(x, 0) = 0$ for $x > 0$
3. $\rho(x, 0) = \rho_{max}$ for $x > 0$, $\rho = 0$ for $x < 0$
4. $\rho(x, 0) = \rho_{max}$ for $x > 0$, $\rho = \rho_0 > 0$ for $x < 0$
5. $\rho(x, 0) = \rho_0$, such that $0 < \rho_0 < \rho_{max}$, for $x < 0$,
   $\rho(x, 0) = \rho_0 + (\rho_{max} - \rho_0)x/L$ for $0 < x < L$, and
   $\rho(x, 0) = \rho_{max}$ for $x > L$.

   What is the difference between the two solutions, if $\rho_{max} = 5$, $u_{max} = 1$ and in the first case $\rho_0 = 1$, while in the second $\rho_0 = 3$?
6. Determine the traffic density that results after an infinite line of stopped cars is started by a red traffic light turning green.
7. At what velocity does the information that the traffic light changed from red to green travel?
8. Calculate the maximum acceleration of a car that starts approximately one car-length behind a traffic light (i.e. $x(0) = -1/\rho_{max}$).
9. Uniform traffic with $\rho(x, 0) = \rho_0 < \rho_{max}$, was suddenly stopped with a red light at $x = 0$ and $t = 0$. (Attn: there will be two shocks).

1.2 Statement: Envelopes and caustics

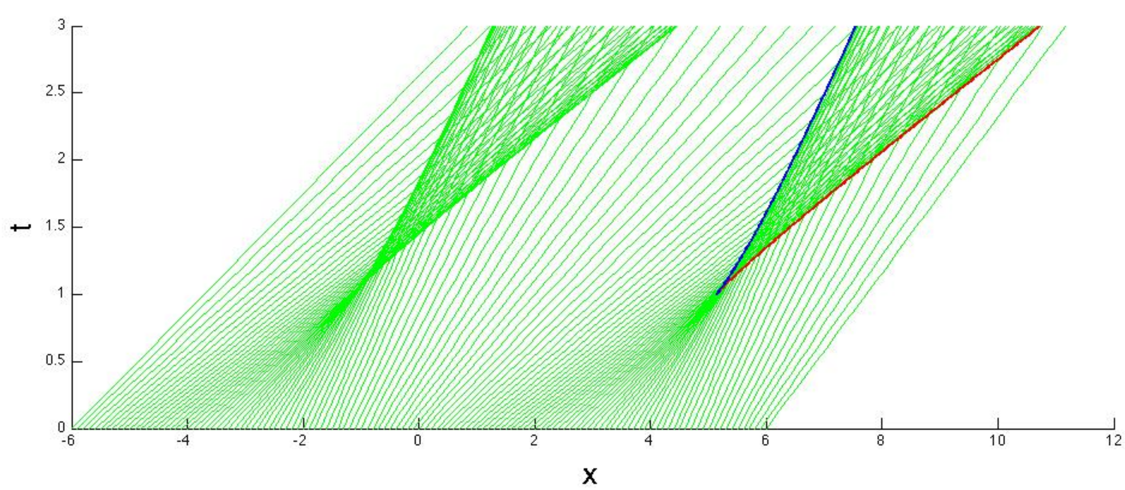
If you were not allowing the shocks to form, there would be regions of multivalued solutions.

For this problem consider inviscid Burger's equation

$$u_t + uu_x = 0$$

with initial data $u = 2 + \sin(x)$ at time $t = 0$.

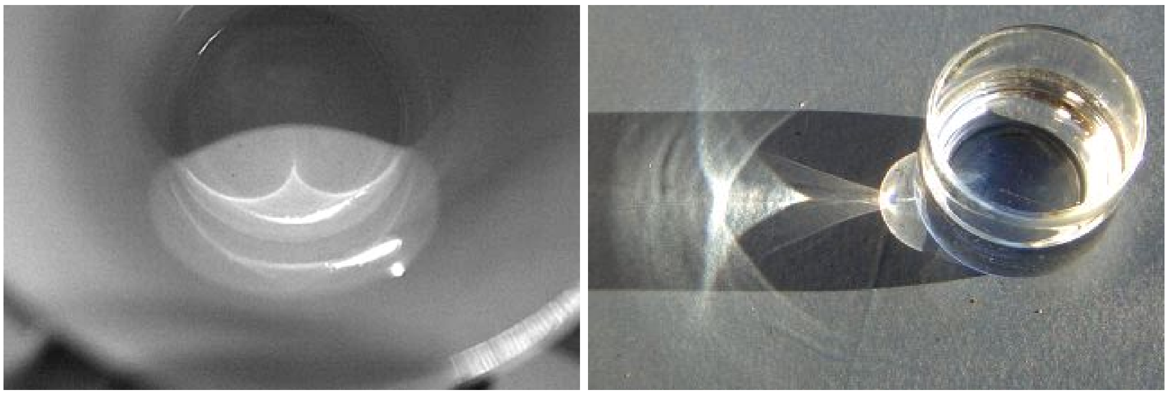
The characteristics look as follows:



Assignment:

1. Find the characteristic curves analytically.
2. When and where does the shock form? What is the speed of the shocks?
3. Plot the characteristics in Matlab, obtain the same picture as above.
4. Find the two envelopes of characteristics analytically. Plot them in MATLAB (here they are the blue and the red curves, and they form a "cusp"). A similar phenomenon in optics, when the rays are rays of light, is called a *caustic* (which is the envelope of the light rays). Below is a picture of a caustic in a cup and produced by a glass of water (images from wiki).
5. Compute analytically the slope **between** the two branches of the envelope at the point where they start (i.e. at the time when a shock would form). The slope that you will obtain is the same for all problems.

Example of a caustic in a cup and a caustic produced by a glass of water:



2 Special problems

2.1 Statement: Eikonal equation

The Eikonal equation (from German *eikonal*, from Greek *image*) is a non-linear PDE encountered in problems of wave propagation, geometric optics, etc.

$$|\nabla u| = f(\mathbf{x}), \quad i.e. \quad in \quad 2D \quad \sqrt{u_x^2 + u_y^2} = f(x, y)$$

The solution for $f(x, y) = 1$ gives a signed distance function from the given initial profile $u(x_0, y_0) = 0$ profile.

Assignment:

1. Find the position of the front of the wave at time t = R, which starts at $(x_0, y_0) = (x_0, x_0^2)$ by finding an envelope of circles located at (x_0, y_0) with radius R. Plot it in Matlab.
2. Use the method of characteristics for a generic first order PDE to solve this problem with the given initial profile $u(x, y) = 0$ on the curve $y = x^2$. The first steps are:
   - Introduce the new variables $u_x = p$ and $u_y = q$
   - introduce a function $H = p^2 + q^2 - 1 = 0$.
   - differentiating this equation with respect to x and y, you get a "system" of two equations
   - in the new system introduce characteristic speed (in 2D!) with components dx/dt and dy/dt
   - proceed to solve the problem
3. How should you scale the time along the characteristics so that at time t the distance to the initial front was t? In other words, how does t of your choice relate to R?
4. Use Matlab to plot on the same graph the solution at the rescaled times t = 0, 0.1, and 0.3. When does the swallow tail first form?