18.311 — MIT (Spring 2012)

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Problem Set # 01.

Due: Friday February 24.

Turn it in before 3:00 PM, in the box provided in Room 2-108.

IMPORTANT: The Regular and the Special Problems must be **stapled in TWO SEPARATE packages,** each with your **FULL NAME** clearly spelled.

Contents

1 Regular Problems.

1.1 Statement: Dimensional analysis and diffusion speed.

In the lectures it was shown that if $\Theta = \Theta(\vec{x}, t)$ denotes the concentration of salt in water (e.g.: grams per liter) then, **assuming that there is no motion** by the water

$$\Theta_t = \nu \, \Delta \, \Theta, \tag{1.1}$$

where ν is the diffusion coefficient — which we assume here to be constant[1] — and Δ is the *Laplacian operator*, $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$. The **same equation** (with a different value of ν) applies if, for example, Θ denotes sugar concentration, or the concentration of some coloring (e.g. ink).

Under the conditions where (1.1) applies, imagine that you inject a very tiny blob of ink inside the liquid.[2] Then the size of the blob will start increasing with time (due to the ink's diffusion). The blob's edge will cease to be sharp as time goes on, but here we will simplify things and assume that they remain sharp enough during the course of the experiment.

Make the approximation that, at time $t = 0$ (when you start the experiment) the blob is just a point. Then, using qualitative and physical arguments only **(no solution of the equation)**

1. Argue that the shape of the blob is a sphere for $t > 0$.

2. Find a formula for the radius of the blob $R = R(t)$ as a function of time. There is a numerical multiplicative constant α, as in $R = \alpha\, f(t)$, which you will not be able to determine (without solving the p.d.e.), but you should be able to get $f(t)$.

Typical values for diffusion coefficients, as in the examples above, are $\nu = \gamma\, 10^{-5}\ \mathbf{cm^2/sec}$, where γ ranges[3] between $1/2$ and 2. **Assume that $\alpha = \gamma = 1$.** Then

3. What is the radius of the blob when $t = 1$ minute?

4. At what time does $R = 5$ cm?

These numbers should give you an idea of how long it would take to sweeten a cup of coffee if you just deposited a lump of sugar in it, and did not stir the coffee.

1.2 Statement: Dispersive Waves and Modulations.

Consider the following linear partial differential equations for the scalar function $u = u(x,\, t)$:

$$u_t + c\, u_x + d\, u_{xxx} = 0, \tag{1.2}$$
$$u_{tt} - u_{xx} + a\, u = 0, \tag{1.3}$$
$$i\, u_t + b\, u + g\, u_{xx} = 0, \tag{1.4}$$

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[1]Generally ν is not quite constant, since is depends (for example) on the temperature.

[2]Carefully, so as not to start motion. The ink density must match that of water, to avoid gravity induced motion.

[3]Larger molecules diffuse slower than smaller ones; thus $\gamma(\text{sugar}) < \gamma(\text{salt})$. Also, γ grows with temperature, and may even depend on the concentration.

where the equations are written in a-dimensional variables, (c, d, a, b, g) are real constants, and $a > 0$. These equations arise in many applications, but we will not be concerned with these here. It should be clear that, in all three cases,

$$u = Ae^{i\,(k\,x\,-\,\omega\,t\,+\,\theta_0)}, \quad \text{where} \quad \omega = \Omega(k), \tag{1.5}$$

is a solution of the equations, for any real constants $A > 0$, θ_0, k, and ω, provided that

M1. For equation (1.2): $\Omega(k) = c\,k - d\,k^3$ ....**Verify that this is true.**

M2. For equation (1.3): $\Omega(k) = \pm\sqrt{a + k^2}$ .... **Verify that this is true.**

M3. For equation (1.4): $\Omega(k) = -b + g\,k^2$ .... **Verify that this is true.**

Remark 1.1 *Solutions such as that in (1.5) represent monochromatic sinusoidal traveling waves, with* **amplitude A**, **phase $\theta = k\,x - \omega\,t + \theta_0$**, **wave number k**, *and* **angular frequency ω**. *The* **wave length** *and* **wave period** *are $\lambda = 2\,\pi/k$ and $\tau = 2\,\pi/\omega$, respectively. The wave profile's crests and troughs move at the speed given by $\theta = $ constant, namely: the* **phase speed $c_p = \omega/k$**.

Remark 1.2 *In all three cases, $\Omega = \Omega(k)$* **is a real valued function of k**, *with $\dfrac{d^2\Omega}{dk^2} \neq 0$ — i.e.: Ω is not a linear function of k. Because of this, we say that the* **equations are dispersive** *and call Ω* **the dispersion function.** *The (non-constant) velocity $c_g = c_g(k) = d\omega/dk$ is called the* **group speed**, *and the objective of this problem is to find out what the meaning of c_g is.*

The reason for the name "dispersive" is as follows: In a dispersive system, waves with different wavelengths propagate at different speeds. Thus, a localized initial disturbance, made up of many modes of different wavelengths, will **disperse** *in time, as the waves cease to add up in the proper phases to guarantee a localized solution: Localization depends on destructive interference, outside some small region, of all the modes $a(k)\,e^{i\,(k\,x+\theta_0)}$ making up the initial disturbance. However, since these modes propagate at different speeds, the phase coherence needed for the destructive interference is destroyed by the time evolution.* **This phenomena is illustrated by figure** 1.1**.**

1.2.1   Tasks to be performed.

TASK 1. verify **M1** through **M3**, above below equation (1.5).

TASK 2: Consider a dispersive waves system, that is: a system of equations accepting monochromatic traveling waves as solutions, provided that their wave number k and angular frequency ω are related by a dispersion relation
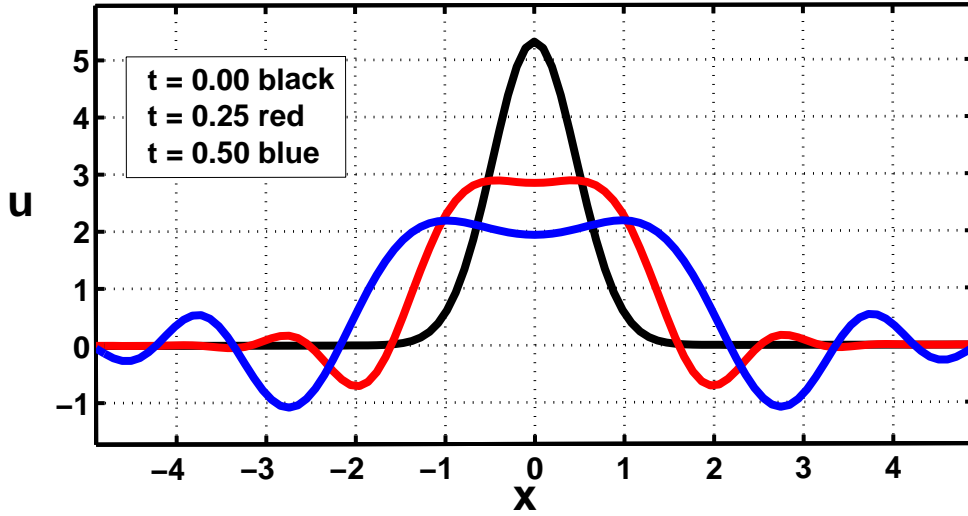
$$\omega = \Omega(k). \tag{1.6}$$

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Figure 1.1: Example of dispersion: initial "Gaussian" bump, as it evolves under a dispersive equation with $\Omega(k) = \pm k^2$ — i.e.: $u_{tt} + u_{xxxx} = 0$. Solution at times $t = 0, 1/4, 1/2$ displayed. As the initial lump's phase coherence is destroyed by dispersion, localization is lost, and the bump "disperses".

Consider now a slowly varying, nearly monochromatic solution of the system. To be more precise: consider a solution such that at each point in space–time one can associate a *local wave number* $k = k(x, t)$ and a *local angular frequency* $\omega = \omega(x, t)$. In particular, assume that both k and ω vary slowly in space and time, so that they change very little over a few wavelengths or a few wave periods — *on the other hand, they may change considerably over many wavelengths or wave periods.* Then

Assuming conservation of wave crests, derive equations governing k and ω.

These equations are called the **Wave Modulation Equations.**

Remark 1.3 *Notice that the assumption that k and ω vary slowly is fundamental in making sense of the notion of a locally monochromatic wave. To even define a wave number or an angular frequency, the wave must look approximately monochromatic over several wavelengths and periods.*

Remark 1.4 **Why is it reasonable to assume that the wave crests are conserved?** *The idea behind this is that, for a wave crest to disappear (or for a new wave crest to appear), something pretty drastic has to happen in the wave field. This is not compatible with the assumption of slow variation. It does not mean that it cannot happen, just that it will happen in circumstances where*

the assumption of slow variation is invalid. There are some pretty interesting research problems in pattern formation that are related to this point.

Hint 1.1 *It should be clear that one of the equations is* $\omega = \Omega(k)$, *since the solution behaves locally like a monochromatic wave. For the second equation, express the density of wave crests (and its flux) in terms of* k *and* ω — *in a sinusoidal wave:* How many wave crests are there per unit length? How many wave crests pass through a fixed point in space per unit time? *Then write the equation for the conservation of wave crests using these quantities.*

1.3 Statement: Exercise in implicit differentiation #01.

In each case compute $\dfrac{dy}{dx}$ as a function of y and x, given that $y = y(x)$ satisfies:

1. $x^3 + xy + 2 = 0$.

2. $y = \sin(y + x)$.

3. $\ln(y) = x$.

4. $\cos^2(y) = x$, for $x > 0$.

5. $y = f(c - y\,x)$, where f is an arbitrary function and c is a constant.

6. $y = f(x - c\,y)$, where f is an arbitrary function and c is a constant.

2 Special Problems.

2.1 Statement: Chromatography problem.

In chromatography, and similar exchange processes studied in chemical engineering, the following situation arises:

A fluid carrying dissolved substances (or particles, or ions) flows through a fixed bed, and the material being carried is partially absorbed on the fixed solid material in the bed. Let the **fluid flow** be idealized to have a **constant velocity** V. Let ρ_f be the **density of the material carried in the fluid**, and ρ_s be the **density deposited in the solid.** The amount of material being deposited can be related to the amount of material in the fluid by the **exchange equation**, which in its simplest form can be taken to be:

$$(\rho_s)_t = k_1 \left(A - \rho_s\right)\rho_f - k_2\,\rho_s \left(B - \rho_f\right), \tag{2.7}$$

where k_1, k_2, A, and B are constants. The *first term* in this equation represents the deposition from the fluid to the solid, at a rate proportional to the amount in the fluid — but limited by the amount already in the solid, up to a capacity A. The *second term* represents the reverse transfer from the solid to the fluid, at a rate proportional to the amount in the solid — but limited by the amount already in the fluid, up to a capacity B.

At equilibrium $(\rho_s)_t = 0$, and ρ_s is a definite function of ρ_f. In slowly varying conditions, which will arise when the **reaction rates k_1 and k_2 are "large"**, we may still take the approximation in which $(\rho_s)_t = 0$ as far as equation (2.7) is concerned — i.e.: **quasi-equilibrium.**

Remark 2.5 *As usual, the idea here is that, compared with the other time scales in the problem, the reactions leading to equation (2.7) are very fast. Thus, any deviations from the equality relating ρ_s and ρ_f*

$$k_1(A - \rho_s)\rho_f - k_2\rho_s(B - \rho_f) = 0\,, \tag{2.8}$$

that occurs when $(\rho_s)_t = 0$ in (2.7), are very rapidly damped out (so that, at any time, we can assume that the equality above is satisfied.)

Your task: Assuming the quasi-equilibrium approximation, and using the conservation of the material being exchanged, derive an equation for $\rho_s = \rho_s(x,\, t)$.

Hint 2.2 *The total density of the material is $\rho = \rho_s + \rho_f$, while the flow rate follows from knowledge of ρ_f and the (constant) fluid velocity V.*

THE END.